



GRADUATE SCHOOL OF BUSINESS

DEDUCTIVE NECESSITY AND THE LOGICAL
STRUCTURES OF REASONING: PIAGET'S
PSYCHO-LOGICO MODELS

DR. ROBERT H. SEIDMAN
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DEDUCTIVE NECESSITY AND THE LOGICAL STRUCTURES OF REASONING:
PIAGET'S PSYCHO-LOGIC MODELS

A. Introduction

In one of his popular explications of relativity theory, Albert Einstein notes that Euclidian geometry, like all geometries, is concerned not at all with the relationship of its ideas and entities to the objects of our experience. On the contrary, geometry is concerned only with the logical connection of these ideas and entities amongst themselves. This is, of course, true of all pure mathematics.

Furthermore, while it is apparently meaningless to assert the truth of a mathematical system's axioms, we are nonetheless inclined to accept them as given. In Einstein's words:

Then, on the basis of a logical process, the justification of which we feel ourselves compelled to admit, all remaining propositions are shown to follow from those axioms, i.e. they are proven. A proposition is then correct ('true') when it has been derived in the recognized manner from the axioms.
 (Einstein, 2)

Einstein thus raises a fundamental question: what is the nature of this "logical process," this "recognized manner," whose justification, Einstein and all of us apparently feel "compelled to admit?" Perhaps its nature can be best captured by the terms logical necessity and self-evidence. Is it derived from our physical world, intrinsic in the laws of nature and thus discoverable in the same sense as pi mesons are? Or is this notion merely an invention (a construct) like the relativistic space-time continuum and, like modern physics, subject to continual revisions and reconstructions? How does this notion evolve biogenetically and what is its ontogenesis? I make no attempt in this report to provide a conceptual analysis of "deduction." The interested reader is referred to Suppes (1973), von Wright (1971) and Sosa (1975).

The notions of logical necessity and self-evidence are not confined only to the realms of geometry, logic and pure mathematics. Consider the following items.

Item 1 - Object Permanence. According to studies by Piaget and others, object shape and size become stable concepts at about the eighth month of life. Prior to this state of development, the neonate evidences no searching behavior when a much sought after object disappears from its visual field. However, between the ages of 8-10 months, the child begins to construct this first invariant: permanent object in proximal space. Object permanence is now a self-evident notion.

Item 2 - Conservation of Quantity. If a young child is shown a ball of clay or plasticine that is then split and rolled into two equal balls before his eyes, he will readily admit the equality of the two balls of matter. If, however, one of the balls is rolled into an elongated sausage shape, the child will claim that there is either more or less substance in the new shape, depending on the thickness of the sausage. Similarly, if the sausage is chopped into small pieces, the child will claim that there is more quantity in the collection of the pieces than in the other ball.

If the young child fills two identical glasses with equal volumes of liquid, she will claim that the quantities are no longer equivalent when the contents of one of the glasses is poured into a narrower or wider glass, although she will maintain that it is the same liquid. It is not until approximately the age of 7-8 years that the child is able to perceive the invariance of continuous quantities. Self-evidence of quantity under reversible transformations is apparently not an innate notion and appears to require an intellectual construction.

Item 3 - Transitivity of Length and Weight. If the young child is shown sticks A and B of equal lengths and sticks B and C of equal lengths, he is not at all certain of the equality of sticks A and C unless viewed together. The child is unable to carry out this apparently simple deduction until the age of 7-8 years.

Given two identical brass bars, the young child concedes their equality of weight, $A=B$. She is then asked to compare the weight of B with that of a lead ball, C. Although the child expects C to be heavier, she can see that $B=C$ on a balance. She is then asked whether or not $A=C$ after being reminded that $A=B$ and $B=C$. Until about the age of 8-9 years, the child remains unconvinced of their equality. Apparently, the notion of logical necessity is an ontogenetic one.

Item 4 - Conservation of Discontinuous Quantity and Number. A child places pairs of red and blue beads into two identical glass beakers by first depositing a red bead into Beaker #1 with his left hand and a corresponding blue bead into Beaker #2 with his right hand. The child will readily admit that the beakers contain identical numbers of beads until the contents of one beaker is poured into a narrower or wider beaker. It is not until the age of 7-8 years that the child can conserve number in this sense.

When a 4-5 year old is presented with a row of counters and is asked to construct another row containing the same number of counters, she will typically construct a row similar in length without regard to equivalent cardinality. A 5-6 year old given the same task, uses the notion of one-to-one correspondence to construct a row equal in number and length to the model. However, should this child observe one of the rows lengthened without a change in the number of counters, she declares that the corresponding cardinalities are no longer equivalent. It is not until about 6-7 years that the child is able to conserve number in this sense.

Item 5 - Classification and Seriation. When presented with a box of 20 wooden beads, 2 brown and 18 white, and if the young child concurs that all of the beads are wooden, he is then asked whether there are more wooden or brown beads? Most children reply that there are more brown beads and do not correctly respond until the age of 7-8.

If a young child is asked to place a set of sticks all differing slightly in length, in ascending order, she is unable to systematically construct the series until 7-8 years of age.

If a young child is shown two sticks of slightly different lengths, $A < B$ and then shown $B < C$ while A is hidden, he is unable to make the deduction, $A < C$, until about age 7-8.

Item 6 - The Logic of Verbal Propositions. It is widely believed that Piagetian theory holds that until the age of 11-12 years, the child is unable to reason with simple verbal statements (propositions). This has been interpreted to mean that it is only after the age of 11-12 that the child can "handle" conditional statements such as:

If it rains today, then I will carry my umbrella.

and conditional arguments such as:

If it rains today, then I will carry my umbrella.

It rains today.

Therefore, I will carry my umbrella.

If the above is the case, then as far as conditional statements reflect logical necessity, their acquisition is ontogenic.

Evidence for these Items 1 through 5 can be found in Piaget (1941), Piaget and Inhelder (1941), Piaget (1952), Piaget (1964) and Piaget (1967), respectively.

What appears to be most striking about the preceeding Items is the child's progressive development from an apparent inability to make, what seem to be, the most elementary deductions to a state of awareness that could be characterized as one of self-evidence or logical necessity. As we shall see, the child has, at various stages of his development a certain kind of logic and notion of self-evidence that can perhaps best be characterized as "naive" logic. Just how does the child progress through the transformation to what we call "adult" logic?

Piaget's genetic epistemological approach appears to be a fruitful framework for addressing this question. Piaget has almost single-handedly forged the discipline and it is his approach that has shaped much of developmental and child psychology even though his epistemological viewpoint is not well understood and is not entirely without problems. See critiques in Mischel (1971), in Hamlyn (1978) and in Piatelli-Palmarini (1980), which contains a debate between Piaget and Noam Chomsky. To Piaget, epistemology comes first and psychological investigations second. The former serves as the catalyst and framework for the later.

B. Meta-theoretical Foundations: Genetic Epistemology

1. Genetic Epistemological Goals¹

A prominent goal of genetic epistemology is to offer an explanation of knowledge, especially scientific knowledge,

...on the basis of its history, its sociogenesis, and especially the psychological origins of the notions and operations upon which it [knowledge] is based. (Piaget, 1970b, 1)

¹ The most complete explication may be found in Piaget's as yet untranslated three volume work (Piaget, 1950). My sources are from his translated writings, Piaget (1966; 1968; 1970; 1972a,b,c) and from secondary sources, Boyle (1969); Flavell (1963); Mays (1953).

If knowledge is viewed as a continuous construction of reality, then the major goal of genetic epistemology is to investigate this growth of knowledge while avoiding the pitfall of becoming fixated upon one or another of its static structures. Thus, particular attention is paid to the transformation from adequate but relatively poorer kinds of knowledge to states that are apparently richer in both intention and extension. Einstein's contribution to the phenomenon of relativity did not completely destroy previous paradigms, but made them less adequate to describe reality. These inadequate paradigms were transformed, incorporated and enriched by the new structures (see Kuhn, 1970). Genetic epistemology studies the growth of knowledge both socio-historically and ontogenetically and tries to illuminate the fundamental connections. I will focus upon the ontogenesis of knowledge. The socio-historical connections may be found in Piaget (1952; 1966).

If we postulate that knowledge is in a continual state of ongoing development characterized by the progression from one state to a more complete and effective one, then to accurately analyze this process genetic epistemological research must employ suitable methods and tools. A collaboration between psychologists and logicians has proven fruitful.

2. Logic and Psychology

Epistemology is the study of valid knowledge (i.e., how we come to know reality) and genetic epistemology studies the process which is the passage from lesser to greater validity. This raises the question of how our knowledge reaches reality and with it the relation between fact and validity, subject and object. If the question was one of pure validity, epistemology would depend solely upon logic. On the other hand, if the question was one of merely fact, epistemology would reduce to a psychology of cognitive functions.

Thus, if psychology is incompetent to describe forms of validity, and if logic is incapable of discovering empirical facts, a synthesis is essential in order to make sense out of the questions that epistemology raises. Experimental psychology is necessary to verify questions of fact that arise and logic is necessary to not only place these facts in some coherent and meaningful context for description and analysis, but to judge questions of validity. We may then view one of psychology's roles as studying subjects of all ages who adopt norms that logic must verify.

It is within the logician's purview to formalize the structures that are suitable to describe the successive stages of norm development. One of the logician's roles is to determine the value of these norms and the characteristics of the epistemic progression or regression shown by the subject's cognitive development as studied by the psychologist. Thus, we are better able to perceive the subject's dual nature: that of psychological subject and that of epistemic subject, with the latter carrying within it the seeds of development of the epistemic norms that we wish to formalize and study.

The subject who, for example, comes to recognize the logical necessity of length transitivity at 7-8 years clearly develops a norm. But how does this development occur and why? Can we be sure that it is acquired through experience alone? Is it transmitted from adult to child? Does it result from language and semiotic or symbolic construction? Piaget suggests that such norms arise as

...the product of a partly endogenous structuration and proceed by equilibration or progressive autoregulation. (Piaget, 1972,9)

According to this view, knowledge arises not from a self-conscious subject, nor does it arise from objects already constituted which impress themselves upon the knower. Knowledge arises from the

interactions that occur between the two. Thus, the initial problem of knowledge is the construction of intermediaries or structures that start from the point of contact between the body and the external world of things. It is action and not perception alone which plays the mediating role in the construction of epistemic reality.

It is these structures, their development and transition that allow the child to construct the world of knowledge (and in particular, logico-mathematical knowledge) and leads to varying forms and developmental stages of logico-mathematical self-evidence and logico-mathematical necessity (hereafter, "self-evidence" and "logical necessity").

It is Piaget's hypothesis that there is a correspondence between psychological formation and logical formalization. Piaget does recognize that there are limitations to logical formalizations: 1) any one logic by itself is inadequate and all logics taken together are too rich to enable logic in and of itself to form a single value basis for knowledge; 2) Godel's incompleteness theorems (Godel, 1931; Nagel and Newman, 1958) demonstrates the inherent limitations to formalization and axiomatization; 3) knowledge is not purely formal.

Despite these limitations, logical formalization is a powerful descriptive tool for the study of the structures of knowledge. Although one task of genetic epistemology is to study the nature of the transitions (this is a factual question) from lower to higher states of knowledge, Piaget would leave it to the specialists to determine which state is more advanced and to the psychologists to obtain the facts involved.

3. Ontogenesis of Logico-Mathematical Knowledge

What is logico-mathematical knowledge and how does it arise? This knowledge, according to Piaget's theory, is gained through a process called reflective abstraction which is derived from the fundamental process of organism adaptation.

Adaptation is a process that consists of the complementary processes of accommodation and assimilation, and is self-regulated by an equilibration mechanism. The nature of these processes is succinctly summarized by Elkind:

[Piaget] argues that intelligence is an extension of biological adaptation which, in lieu of the instinctive adaptations in animals, permits relatively autonomous adaptations which bear the stamp not only of our genetic endowment, but also of our physical and social experience. On the plane of intelligence we inherit the processes of assimilation (processes responsive to inner promptings) and of accommodation (processes responsive to environmental intrusions). Assimilative processes guarantee that intelligence will not be limited to passively copying reality, while accommodative processes insure that intelligence will not construct representations of reality which have no correspondence with the real world. (Elkind, 1969, 329)

The activities of imitation and play serve nicely to illustrate the concepts of accommodation and assimilation, respectively. Imitation is, for the most part, independent of internal forces and is responsive to the influences of the environment. On the other hand, play is mostly responsive to internal forces and somewhat independent of environmental influences.

It is intelligence which maintains an equilibrium between assimilative and accommodative activities and is therefore "relatively autonomous" of both internal and environmental forces. For example, the conclusion of a deductive argument is a new piece of information. An adaptation has taken place. Here we have an assimilation but no object

transformation (the premises have not been altered) and we have an accommodation without the alteration of any mental structures (the reasoning processes have not been modified).

Elkind, sums it well:

Reason, or intelligence, is thus the only system of mental processes which guarantees that the mind and the environment will each retain its integrity in the course of their interaction The question is not how much nature and nurture contribute to mental ability, but rather the extent to which various mental processes are relatively autonomous from environmental and instinctual influence.

.....
Those processes which show the greatest independence from environmental and internal regulation, the rational processes, are the most advanced of all human abilities. It is for this reason that Piaget reserves for them, and for them alone, the term intelligence. (ibid., 330)

Ontogenesis is usually framed, when described in Piagetian terms, within the context of the assimilation-accommodation-equilibration processes. I have not adopted that approach here because the biological metaphor is not as well suited as the logical metaphor (reflective abstraction) to the description of the processes examined in this study. These processes, the rational processes, include those components of intelligence that we call deductive and as such are the fundamental components of intelligence and understandably one of the most difficult to describe. For a thorough and deep explication of the biological metaphor associated with adaptation, see Piaget (1966).

Before we can consider the development of logico-mathematical structures and developmental stages we must determine just how logico-mathematical knowledge comes about. Only then will we be able to understand the development of logico-mathematical reasoning.

Let us begin by agreeing that logical and mathematical structures and knowledge are abstract and that physical knowledge is concrete (it is based upon direct experience). But what is logical and mathematical

knowledge abstracted from? Piaget finds it useful to make the distinction between simple (or empirical) abstraction and reflective abstraction. Experimental or empirical knowledge appears to be derived from objects themselves, through perception and sensorial origins.

However, Piaget postulates that

... our knowledge stems neither from sensation or from perceptions alone but from the entire action, of which perception merely constitutes the function of signalization. The characteristic of intelligence is not to contemplate but to 'transform' and its mechanism is essentially operatory. (Piaget, 1972c, 67)

Operations are interiorized actions that are coordinated into group-like structures. Thus, we can only know an object by acting upon it and transforming it in some way just as the organism adapts to the world through the processes of assimilation and accommodation through the mechanism of auto-regulation.

One way of transforming the object we wish to know is by modifying its position, its movement, or its characteristics in order to explore its nature. This is, of course, physical action. Another way of transforming the object is by enriching it with characteristics or new relationships which retain the object's original characteristics or previous relationships but at the same time complete them by systems of, for instance, classification, numerical order and measure. These are logico-mathematical actions. Thus, we find that logico-mathematical abstraction is derived from the action itself and not from the object acted upon. Two examples will help to clarify the distinction between the two kinds of abstraction and to illustrate the notions of logico-mathematical knowledge and necessity.

Case 1: Empirical Abstraction. A child is able to heft objects in her hands and thus can come to the realization that they have different weights. Sometimes bigger things weigh more but other times smaller things are heavier. She finds all of this out experimentally, thus her knowledge is abstracted from the objects themselves.

Case 2: Reflective Abstraction. Many young children can count up from one to ten and thus assert that there are ten pebbles lined up in a row. But their conception of number is shakey as we saw in Item 4. Piaget frequently uses the following illustration to give concrete meaning to reflective abstraction.

A small child was engaged in counting pebbles and after lining them up in a row proceeded to count them from left to right and counted ten. He then counted them from right to left and again counted ten. No matter how he arranged the pebbles the child always counted ten. The child thus discovered the notion of mathematical commutativity through his actions. Commutativity is not an intrinsic property of pebbles in and of themselves, although there certainly was a physical aspect to the knowledge involved. Order was not inherent in the pebbles themselves, it was the subject who arranged them in different configurations. Thus, the logico-mathematical notion of commutativity was derived not from the mere physical properties of the pebbles but from the actions of the child upon the pebbles.

Logico-mathematical experience consists of acting upon objects. The abstraction of knowledge is based, however, on actions and not solely upon the objects themselves. Action begins by conferring upon objects certain characteristics that they previously did not have but which enable them to retain their intrinsic characteristics. The experience is concerned with the connections between characteristics introduced by actions on the object.

What the child discovers in Case 2 is not a physical characteristic of pebbles, but an independent relation between the two actions of reunion and ordination. This experience is authentically logico-mathematical, since it deals with the actions of the subject and not with objects per se. At a certain developmental level, according to Piaget's theory, actions can be interiorized as symbolically manipulated operations thus avoiding the need to manipulate physical objects.

The term "reflective" has two psychological senses, according to Piaget. In its first sense, reflection is the transformation from one hierarchical level to another. For example, the transition from the level of action to the level of operation (internalized action). In its second sense, reflection refers to the process of mental reflection, meaning that at the level of mental thought a reorganization takes place. A general model of equilibration, which includes reflective abstraction, can be found in Piaget (1976, 1977).

For Piaget, action and operation do not appear as singular entities standing alone and reflective abstraction is not based upon action but upon coordinated action:

Actions can be coordinated in a number of different ways. They can be joined together, for instance; we can call this an additive coordination. Or they can succeed each other in temporal order; we call this ordinal or a sequential coordination Another type of coordination among actions is setting up a correspondence between one action and another. A fourth form is the establishment of interactions among actions. (Piaget, 1970b, 18)

Piaget contends that the various forms of coordinated actions (schemes) have parallels in coordinated logical operational structures. The coordinations at the level of action are thus the bases of the higher level psychological structures as they develop in mental thought. These operational structures can be described by formalized logical structures.

It is important to note that Piaget does not postulate that the formation of logico-mathematical structures is explained by language alone, although language coordinations are indeed important. These more advanced logical structures are formed by the coordination of actions which are the bases of reflective which is in turn the fundamental

process involved in the ontogenesis of intellectual structures and hence of knowledge itself.²

4. Reflective Abstraction and Logical Necessity

We have seen how a certain arithmetic structure (commutativity) is constructed through reflective abstraction. While some logic acquires a degree of necessity when the child reaches the age of 7-8 years, this logic is the result of the progressive building up of logical operations. Piaget has demonstrated (Inhelder and Piaget, 1958 - hereafter, Piaget, 1958) that the ontogenesis of logico-mathematical structures in the child develops gradually although sometimes logical necessity and self-evidence crystalize rather suddenly. We will see later that portions of adult logic crystalizes at age 7-8 and at 11-12 years when the structures of concrete and formal operations materialize, respectively. According to Piaget, two complementary reasons explain the crystalization of logical necessity.

The first reason has to do with the closure of operational structures. Let us re-examine one of the examples in Item 5. When a young child is presented with a number of sticks all differing slightly in length, say, $A < B < C < \dots$, he will typically attempt to order them serially by merely groping with no systematic method employed. Since this groping is of an empirical kind the logical structure cannot be said to be closed. Thus, transitivity applied to some of these objects ($A < C$ if, $A < B$ and $A < C$) appears not to be necessary, but merely possible or probable.

² Piaget readily admits the pitfalls of an infinite regress beyond the coordination in the neuron network and into the organic coordination. He wants only to carry the regression back into its beginnings in the psychological organism. We thus view the beginnings of logical structures in psychology and not in biology. For a detailed explication of this matter see Piaget (1971, 1972).

However, at about the age of 7-8 years, seriation is established operationally by systematically choosing the smallest remaining element in the series. This is an indication that the child realizes that element C is both smaller than elements D, E and those that follow and is bigger than A, B and C. Thus, the structure involved becomes whole and closed, and relations within this structure are now interdependent and are able to be composed amongst themselves without any recourse to things outside of the system.

At this stage in the child's intellectual growth, transitivity appears as a necessity and as Piaget puts it:

[L]ogical 'necessity' is recognized not only by some inner feeling, which cannot be proved, but by the intellectual behavior of the subject, who uses the newly mastered deductive instrument with confidence and discipline. (Piaget, 1971, 316)

The second reason accounts for the formation and closure of the relevant structure. A structure, according to Piaget, can impose itself as a necessity and does this by "endogenous" means. This structure is the product of a "progressive equilibration."

The necessary character of logico-mathematical structures emerges from their "progressive equilibration" which is in turn derived from organism autoregluation. Piaget notes that the hereditary character of instinct, for example, precludes both generality and necessity because it is species specific and wholly contingent in nature. Instinctive behavior is always of a particular and specialized kind compared to the "mobile universality" of intelligence. Internal equilibration within cognitive structures explains the generality and mobility of intelligence through cognitive thought.

New structures are constructed from a kind of endogenous evolution that progresses in stages by autoregulated equilibration and reflective abstraction. Piaget frequently uses results from Godel's incompleteness theorems as an analogy for structural development. Godel has shown that a system which is sufficient for its own purposes cannot by its own means (or by weaker means) verify its own noncontradiction. To do this one must extend the system (make it stronger). From this we see that:

... the development of a structure cannot be made entirely on its own level, by mere extension of given operations and combination of known elements; the progress made consists of the construction of a wider structure, embracing the former but introducing new elements. (ibid., 319).

For example, Cantor succeeded in constructing transfinite arithmetic by generalizing the operation of correspondences. Transfinite arithmetic is derived from elementary arithmetic by abstracting from its results an operation which allows us to construct a new structure but which also includes the old one. Thus, it is not merely the extension or generalization of an old structure to a new one. The noncontradiction of the old structure is now assured by the new structure but the new structure is not capable of assuring its own noncontradiction.

The above logical abstraction process is similar to the process of reflective abstraction. The subject first takes note of the existence of an action or operation. Next this action or operation is reflected or projected into another plane, for example, from the plane of action to the plane of thought or from concrete thought to abstract systemization. Finally, this reflected action or operation is integrated into a new structure. This new structure must subsume the previous one at a higher level of abstraction. The new structure may be established, according to Piaget's theory, only if these two conditions are fulfilled:

(a) the new structure must first of all be a reconstruction of the preceeding one if it is not to lack coherence and congruity; (b) it must also, however, widen the scope of the preceding one, making it general by combining it with the elements proper to the new plane of thought, otherwise there will be nothing new about it. (ibid., 320)

Thus, the construction of new logico-mathematical structures (and all cognitive structures) is a product of new combinations proceeding by means of reflective abstraction. Certain of these structures create the self-evidence and logical necessity of adult logic.

5. The Stages of Logic Ontogenesis

Piaget divides ontogenesis into four stages (see Table 1) which correspond to intellectual structures and arise by the processes described above. While the age levels for each stage are only approximations, the order of the stages are fixed and more advanced stages must be built upon prior structures. The structures representing Stages I and II are of a physical nature and it is not until Stages III and IV that we can talk of abstract logico-mathematical structures.

TABLE 1

STAGES OF INTELLECTUAL DEVELOPMENT

| <u>Stage</u> | <u>Name</u> | <u>Age</u> |
|--------------|----------------------------------|------------|
| I | Sensori-motor Stage | 0-2 years |
| II | Pre-operational Thought Stage | 2-7 |
| | 1. pre-conceptual substage | (2-4) |
| | 2. substage of intuitive thought | (4-7) |
| III | Concrete Operations Stage | 7-12 |
| IV | Formal Operations Stage | 12 onward |

a. The Sensori-motor Stage

At first the young infant's primitive universe does not include permanent objects nor is there any really fixed boundary between self and other. This stage might well be called the "construction of self." Piaget contends that a reality structure that does not include boundaries between subject and objects contains only one possible link which will later serve to differentiate the child's world into subject and objects. This link is action.

The young infant, although not conscious of self as a separate entity, nevertheless relates everything to his own body. Because these actions lack coordination each action constitutes a small yet isolable whole which has the effect of relating the body itself to the object. Since these actions are solitary and isolable, their only reference is to the body itself which in turn causes an automatic centering which is neither voluntary nor conscious. This centering is termed egocentrism. Examples of egocentric actions are sucking, prehension and looking.

The child affirms himself later by freely coordinating his actions. Objects are then constituted as they comply with or resist the coordinations of movement. As the infant progresses through the sensori-motor stage four types of assimilation can be distinguished.

1. Reproductive (or Functional) Assimilation: Pleasurable action induces further action of the same kind (e.g., sucking). The exercise of schemata is inherently satisfying.

2. Generalizing Assimilation: When the completion of a response pattern becomes the stimulus for the repetition of the response we have a circular reaction. During the repetition of a circular reaction the infant may come into contact with a different aliment (i.e., objects that stimulate the assimilatory process) and learn to perform the response upon them. For example, the child learns that he cannot only suck his mother's nipple but also his thumb and the nipple of a feeding bottle.

3. Recognitory Assimilation: The infant shows that she can recognize and discriminate objects by behaving differently towards them in different circumstances. She will reject her thumb in favor of the nourishment of the nipple, for example.

4. Reciprocal Assimilation: This is the very significant coordination of schemata which occurs when schema assimilate each other. Eventually the child will learn to use his looking schema, for example, in order to utilize his handling schema and his prehension schema in order to gain a better view of an object.

There occurs, in this first stage of development, what Piaget terms a kind of "Copernican" revolution. The child's actions are gradually decentered in relation to his own body and his body (i.e., self) comes to be considered as one object among many others in a space. The child now begins to be aware of himself as the source of his actions. The coordination of actions (reciprocal assimilation) tends to displace objects and the subsequent coordination of these displacements elaborates the "group of displacements" and allows the child to assign positions in his spacio-temporal universe to these objects, thus facilitating the differentiation between subject and objects.

Thus, the coordination of actions is the origin of the differentiation between the subject and objects and is the origin of the decentering process on the plane of physical acts. Piaget claims that the reciprocal assimilation involved in action coordination exemplifies new nonpredetermined features that become "necessary" and as such characterize the development of knowledge.

Piaget identifies six substages within the sensori-motor stage which occur in invariant order.

1. Random and Reflex Action Substage (0-1 month): Here, the neonate's behavior is almost totally autistic. He has an extremely limited contact with the external world (mostly at feeding time) and his automatic reflexes are not yet effected by his environment.

2. Primary Circular Reaction Substage (1-4 months): "Primary" means that the first circular reactions that appear are non-purposive and are a part of the infant's innate behavior patterns (e.g., sucking and grasping). However, the child now no longer responds in a purely reflexive manner and begins to display signs of voluntary behavior. This substage marks the start of the growth of schemata and the operation of recognitory assimilation. Because the infant's actions bring him into closer contact with new objects, generalizing assimilation arises which may slightly modify the child's behavior.

3. Secondary Circular Reaction Substage (4-8 months): Reactions here are more outwardly directed than in the previous substage. The child tries to cause the repetition of interesting phenomena. For instance, if she unsuccessfully tries to grasp a suspended object but manages to touch it, the swinging movement may be novel to her. She may try to reproduce this occurrence (reproductive assimilation). When she sees another suspended object she will assimilate it into the same grasping schemata (recognitive assimilation). When she repeats this action in a new situation the assimilation is a generalizing one.

It is not clear just how much intentionality is evidenced at this substage. The child's actions may be a kind of motor recognition of the object involved (i.e., his movements serve to define the object for him). This could very well be the beginning of the internalization process where actions are internalized to contemplative thought. Imitation is greatly evidenced at this substage but only by means of his own actions that he can observe (i.e., since he cannot see his own tongue, ordinarily, he cannot imitate actions involving it).

4. Coordination of Secondary Schemata Substage (8-12 months): At this point, intentional behavior is in evidence as the child directs his behavior towards a desired outcome. Secondary circular reactions have become differentiated with means clearly subordinated to ends (in Substage 3, means and ends formed part of one secondary reaction). The infant's world begins to evidence some organization. Because the baby can now remove obstacles to reach desired objects, we can say that the child comprehends spatio-temporal space. He can now imitate even if he cannot see his own actions and he has a limited ability to apply his schemata to new events. For example, if a doll is hidden behind a cushion, the baby is able to push the cushion aside to find the doll. After some repetitive behavior, if the doll is removed from behind the cushion in full view of the baby, he will still look behind the cushion in search of the doll.

5. Tertiary Circular Reactions Substage (12-18 months): Here, the child in addition to differentiating actions into means and ends, discovers new means. Instead of pushing away a cushion to get at an object, she might pull it away and repeat this action on other occasions (circular). The child now searches for the doll elsewhere if it has been removed from behind a cushion in full view. She is, however, incapable of reasoning about hidden movements. If, for instance, we first place the doll behind a cushion and then surreptitiously remove it, the child will be incredulous when she discovers that the doll is missing.

The baby's imitation powers are more advanced at this substage. She learns about herself and her world through imitation (which is essentially accommodation) and play (which is essentially assimilation).

6. New Means Through Mental Combinations Substage (18-24 months):

The baby now has a mental representation of his world and knows his place in it. He is an object amongst others and his behavior is becoming increasingly emancipated from sensori-motor actions. We see the beginning of the use of symbolization in problem-solving. Imitation can now be deferred.

We should note that the process of reciprocal assimilation can consist of assimilating the same alimnt into two new schemes simultaneously. If, for example, an object shaken makes a noise, it can then become something to listen to and something to look at. This leads to a reciprocal assimilation which leads to reproductive and generalizing assimilation which results in the child's shaking any toy to discover what noises it makes. We can see that in this case the ends and means are essentially undifferentiated. However, at this substage, the child will set a goal before he is able to aim at it and uses different schemes of assimilation to reach it. For example, he may try to move a hanging string in his crib in order to shake sound-producing toys suspended by the string but which are beyond his reach.

We can see in Substage 6 the beginnings of what has been called reflective abstraction. New combinations are constructed by abstraction upon either objects themselves or by abstraction from schemes of action applied to objects. The latter mirrors reflective abstraction. Thus, the child's recognition of a suspended object as something that can be rocked indicates an abstraction of the first kind. The coordination of means and ends while accounting for the proper sequence of movements is a new form of differentiated behavior acquired from actions. This is more akin to reflective abstraction.

Apparently, knowledge, with its "logico-mathematical" and "physical bipolarities" is not structured by language acquisition alone but is formed on the level of action. Actions become coordinated causing the subject and objects to differentiate through the progressive refinement of the mediating structure, called the structured whole. The French term is "ensemble des parties" and its varied connotations are explained in Inhelder and Piaget (1958, ff. 18, xix). These structures are next

interiorized in the form of operations. But before that may happen, the child must pass through another stage of intellectual development.

b. The Pre-operation Stage

While differentiation between subject and objects appear toward the end of the sensori-motor stage, neither seem to have enduring characteristics outside of the present moment. The pre-operational stage of development ushers in a level of conceptualization, however rudimentary. On the level of conceptualized action the subject and object of the action (self or any object) is thought to carry enduring characteristics. This stage may be divided into two substages.

1) The Preconceptual Substage (2-4 years)

Concept formation involves the abstraction of similar features from essentially dissimilar situations and generalizing from particular situations to a general case. Concept formation is closely linked to inductive generalization, but the child at this substage exhibits a logic which Piaget calls transductive: the tendency to link neighboring events on the basis of common individual instances. No common overall property is abstracted by this logic. Transductive logic (also called transductive reasoning) is not limited to the preoperational stage of development. In fact, aspects of transductive reasoning can be seen in all subsequent stages of cognitive development.

Through imitation, the child constructs signifiers of his sensori-motor schemata (the significates). The signifiers are signs (socially shared and need not bear a resemblance to their significates) and symbols (private and bear some resemblance to what they symbolize).

Sensori-motor schemata are thus interiorized as thought and this thought can now be represented by language. This is Piaget's description:

Thus, through the mediation of thought, action is placed in a much larger spatio-temporal context, and raised to a new status as an intermediary between subject and objects. In proportion to the progress of representational thought, the distance between it and its object increases, in time and in space; in other words, the series of successive physical actions, each given momentarily, is completed by representative systems capable of evoking in the form of an almost simultaneous whole, past or future actions or events as well as present ones and spatially distant as well as near ones. (Piaget, 1959, 27)

The child rapidly acquires the capacity to perform elementary inferences, to classify spatial types of configurations and to set up certain correspondences, albeit, very primitive ones. The passage from the realm of action to that of thought is not achieved suddenly. It is the result of a slow and quite laborious differentiation and depends upon certain assimilative transformations. The following items illustrate the kinds of concepts (called preconcepts) that a child in this substage of development might have.

Item 7. A child is given a box containing a large number of wooden shapes. These shapes include red squares and some circles, half of which are red and half as many blue. Also, there are blue triangles in the box. The child is asked to choose from the box four objects that are alike. A typical solution is shown in Figure 1.

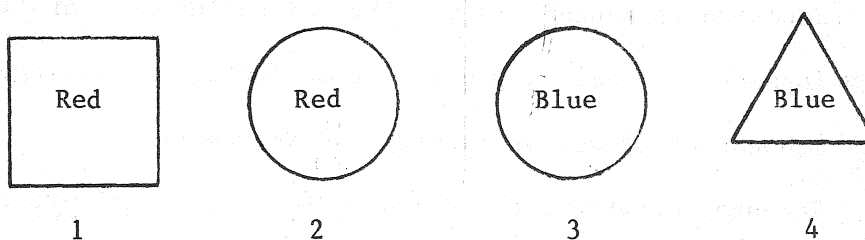


Figure 1. Four Like Objects Chosen Using Transductive Selection

Note that 1 and 2 are linked by color, 2 and 3 by shape and 3 and 4 by color again. No common overall property is abstracted and the selection seems to have been made in terms of properties that are shared by adjacent objects.

Item 8. Piaget, with some humor, tells the story of his young daughter and the slug. While walking one day together, she saw a slug on the ground. Further along on the walk she saw another slug but believed it was the first one reappeared. Even when she returned to view the first slug, then the second, she was unable to express the idea of "another of the same type."

This inability to construct true classes and to work with the idea of inclusion causes the child to make the kind of error shown in Item 5, the wooden bead experiment. In this case, the class wooden beads is an abstraction with subclasses of brown and white beads the reality of the moment. The child concentrates on the most obvious difference between beads, their color. Thus, at this stage of cognitive development, the only intermediaries between subject and objects are pre-concepts and pre-relations. The former lack the quantification of "all" and "some" and the latter lack the relativity of concepts.

Item 9. The child is shown some round red counters and some blue counters. Some of the blue counters are round and some are square. When asked if all round counters are red the child will reply affirmatively, but he will deny that all square counters are blue because there exists round blue counters. Here, the child can identify two classes having the same extension but cannot understand the relation of sub-class. This is because he cannot fully comprehend the notions of "all" and "some," or at least not their operational meanings.

Item 10. If the young child (A) of this substage, has a brother (B) he will readily admit to this fact but will deny that B has himself a brother.

If an object A is to the right of object B, it cannot be to the left of some other object, according to the child at this developmental substage. To the young child, being to the right is an absolute attribute.

For this child, the serial relation $A < B < C$ means that B can only be "in-between" because the quantification "smaller than" excludes that of "larger than."

These pre-concepts and pre-relations, according to Piaget, are intermediaries between schematas and concepts, and are not able to adequately deal with the immediate situation with objectivity. The child's logic is a rather primitive adult logic. Thought is still in the process of developing out of action.

2) The Substage of Intuitive Thought (4-7 years)

Like the previous stage, this substage is characterized by a decentering process and the discovery of certain objective relationships. In the sensori-motor stage of development, centering is focused unconsciously upon the body itself. An analogous centering is reproduced at this substage, but on the higher plane of pre-concepts and pre-relations. In progressing from one structure to the next, what was already evident on the sensori-motor level must now be reconstructed on the higher plane of pre-operational thought.

The decentering at this substage is between concepts (interiorized actions) and like the decentering at the previous stage, is due to progressive coordinations. These coordinations take the form of functions, called constituent functions. Constituent functions are ordinal or qualitative whereas constituted functions (appearing at the next stage) imply "effective" quantification. Both functions exhibit what Piaget calls "univocally straight" application. "Straight" means in the direction of application. An illustration will be useful.

Item 11. If a child of 5-6 years is shown a piece of thread positioned about a peg forming a right angle (see Figure 2) he predicts correctly that pulling one end will lengthen the pulled end and shorten the other end.

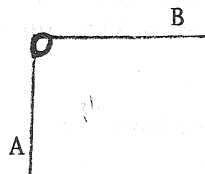


Figure 2. Example Of A Constituent Function

In this Item, one variable is modified because of its functional dependence upon the other and thus because of their coordination, pre-relations become true relations. However, constituent functions lack reversibility, the hallmark of constituted functions and operations. Thus, although the child knows that by pulling segment A, segment B decreases, he is unable to propose the conservation of total length of the string. He lacks the necessary concept of quantification so that the pulled segment is typically assumed to lengthen more than the other segment shortens. The child can thus be said to utilize a "semi-logic" that lacks inverse operations and is not quite yet an operational structure. Constituent functions are highly goal directed and have a strong connection with action schemes.

Because the child's intellectual structures lack this reversibility and lack elementary methods of quantification, there exists no conservation of discontinuous or continuous qualities (see Item 4). The child centers on one aspect or another of the situation in question. For example, the height of a liquid, ignoring its width. As the child progresses through this substage she comes to realize that each change in height is compensated by a change in width. She thus decenters her thinking and is able to think and reason about multiple aspects of the same event simultaneously. The child becomes better able to mentally perform these compensating changes and thus see how she can return to her starting point (reversibility).

Coordinations between conceptualized actions (coordinative assimilations) enables the child at this substage to separate the individual from the class whereas at the preceding substage the child's classification consists of figural collections. A figural collection of

elements consists of elements put together on the basis of relations between disparate things and especially out of the need to give a collection spatial configuration (rows, circles, etc.). These collections are also constructed on the basis of resemblances and differences, but the child has not yet acquired the capacity to separate extension from intension. However, at age 5-6 years the child is able to separate the individual from the class and although collections are no longer figural, the quantification of "all" and "some" is still not achieved. For a detailed treatment of this matter see Inhelder and Piaget (1964). Transitivity of length is still not mastered at this substage (see Item 3).

One final Item will help us view the transition from this stage to the next one.

Item 12. Three colored beads are strung on a wire which is fitted with an opaque cover so that the beads can be slid out of sight. See Figure C-3.

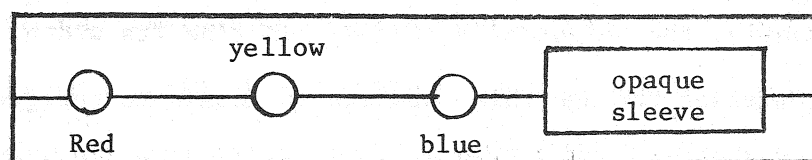


Figure 3. Group Of Transformations Illustration

The beads are pushed into the sleeve as the child watches and he is asked the order of the colors to be sure that he has correctly observed the situation. The frame holding the wire is now rotated 180° and the child is asked which colored bead will emerge from a specified end of the sleeve. The 5 year old will be able to answer correctly for one or two rotations but will be unable to abstract the odd-even rotation relation and the bead order.

This example is particularly instructive because it can be used to illustrate features of a group of transformations. To Piaget, the concept of algebraic group is fundamental to his account of the

operational thought of the next two stages of intellectual development. For no matter how many rotations (partial or full) the frame goes through it will come to rest in a position (state) on the 360° continuum (closure). Any final state can be reached from any starting state by merely rotating the frame clockwise or counterclockwise and the order of rotations is inconsequential (associativity). From any final state we may return to our starting state by performing an inverse rotation. Thus, every action has an inverse. From any state a rotation of plus or minus 360° brings us back to our starting state. A 360° rotation is thus the general identity element. Thus, the states of the frame and the rotations constitute an algebraic group and Piaget contends that the older child's thinking may be described in terms of algebraic groups. The younger child's thought cannot be so described because his semi-logic lacks the operational structure embodying reversibility.

It should be noted that any comprehensive treatment of the above stages and the subsequent ones must take into account the role of language. However, while it is important, it is not crucial for my study of the ontogenesis of logical necessity.

C. Psycho-logic Model I: The Structure of Concrete Operations

1. The Emergence of Concrete Operations

We have seen that the pre-operational child operates on a plane of representation whereas the sensori-motor infant operates on a plane of direct action. The older child, at the concrete-operational stage of development also operates at the plane of representation but at a qualitatively different level.

[The concrete operational child] seems to have at his command a coherent cognitive system with which he organizes and manipulates the world around him. Much more than his younger counterpart, he gives the decided impression of possessing a solid cognitive bedrock, something flexible and plastic and and yet consistent and enduring, with which he can structure the present in terms of the past without undue strain and dislocation, that is, without the everpresent tendency to tumble into the perplexity and contradiction which mark the pre-schooler. (Flavell, 1965).

In the sensori-motor period, cognitive actions are externalized and mostly observable. But as the child gets older, these cognitive actions become more and more interiorized and divested of their concrete substance. The representational cognitive actions cohere to form complex and integrated systems of actions with definitive structural properties.

These tight and highly structured systems are the hallmark of concrete and formal operations. Pre-operations are isolated cognitive expressions which are not part of tight ensembles. Structural systems exist more or less in the sensori-motor and pre-operational stages but it is not until concrete operations that they coalesce into highly structured systems.

Piaget uses logico-mathematical models to describe the cognitive structure of operations (concrete and formal thought). These logico-mathematical structures are models of cognitive structures. The grouping is the model of the concrete operational stage of intellectual development and is the starting point for the subsequent stage.

Structures exist even in the sensori-motor state, according to Piaget, but they lack coherence and are not as well organized as later ones. As an example of early structure, recall that there exists classificatory forms of behavior that occur either in the differentiated state (the subject divides objects into collections) or are inherent in other forms of actions (he acts on objects in some way). The actions

themselves imply classification (e.g., objects which can be sucked and those which cannot). Piaget notes that these "classificatory unities" (his term for pre-classes or "ill-defined classes") form an elementary and imperfect system. In order to understand the nature of a new aliment the infant will seize, shake, suck, and rub it. Thus, the child tries to incorporate it into his own schemes of action.

Multiple structural relations exist between these schemes. For instance, all things that can be grasped can also be seen, but this is not so for the converse. All things that can be heard (and within visual range) can be seen, but the converse is not true. Finally, there are things that can be seen and grasped simultaneously, seen but not grasped, grasped and not seen, and neither seen nor grasped. These systems are a far cry from the coherent structured wholes (capable of closure) of concrete and formal operations. This closure guarantees, as we shall see, the necessity of the combinations that comprise them through direct, reversible and inverse transformations upon operations.

For example, the child at this stage of development is now able to arrange a series of rods differing in size ($A < B < C < \dots$) in order by the exhaustive method of finding the smallest, then finding the smallest of the remaining rods, etc. Now the child recognizes that C will be both greater than A and B ($C > A, B$) and smaller than the remaining rods ($C < D, E, \dots$). The child now takes account of the two relational directions simultaneously. This is an example of reversibility of relations. Because there is an anticipation of the two inverse relations ($>, <$) transitivity comes about (because of the closure of the system) and is viewed now as a necessary relationship.

Conservations (of continuous and discontinuous quantities) are closely connected with transitivity and closure of the structure of concrete operations. Thus, $A=C$ is deemed necessary because $A=B$ and $B=C$ and this is because a property is conserved from A to C. At this stage, once the child accepts $A=B$ and $B=C$ he will view $A=C$ as self-evident by the above arguments.

The child typically uses three types of arguments to support conservation moves: 1) since nothing has been added or removed the quantity is the same. Here we see the appearance of the identity operator of the group of natural numbers; 2) conservation from A to B because A can be restored from B. We see reversibility by inversion; 3) quantity is conserved because the object has simultaneously lengthened and contracted, for example. These two simultaneous modifications compensate one another and illustrates reversibility by reciprocity of relations. Now the child thinks of the system as a whole. The structure is systematic and closed upon itself. As Piaget puts it:

[H]e does not measure in order to evaluate the variations and he only judges their compensation a priori and in a purely deductive fashion, which implies the preliminary postulate of the invariance of the whole system. (Piaget, 1972, 37)

The concrete operational system results from continuous transformations from the previous stage of development (transitions to certain limits mentioned above). These transformations or transitions comprise three interconnected aspects: 1) reflective abstraction derives higher-order structures from lower-order structures; 2) coordination is directed at the whole of the system and by connecting diverse schemes brings about closure; 3) self-regulation brings the systems' connections into equilibrium in their direct and inverse aspects. This equilibrium gives rise to the systems' operational reversibility.

2. Concrete Operational Groupings

Piaget utilizes a mathematical structure called a grouping to model concrete operational cognitive structures. The grouping is a hybrid structure containing components of group and lattice structures. Of the nine distinct groupings (eight major ones and one minor one) which describe this mental structure, four pertain to class operations and four to relational operations. These groupings describe the organization of logical operations (operations dealing with logical relations and classes). In addition, these groupings also serve as models for infralogical operations (Inhelder and Piaget, 1964), values (Piaget, 1941) and interpersonal relationships (Piaget, 1950). The groupings deal with content of the intensive quantification type. Here, relative magnitudes of component parts or subclasses are irrelevant, we need only know whether each part or subclass is less than its whole or supra-ordinate class.

I shall use the zoological class hierarchy in Figure 4 as a paradigm for the explication of the concrete operational groupings.

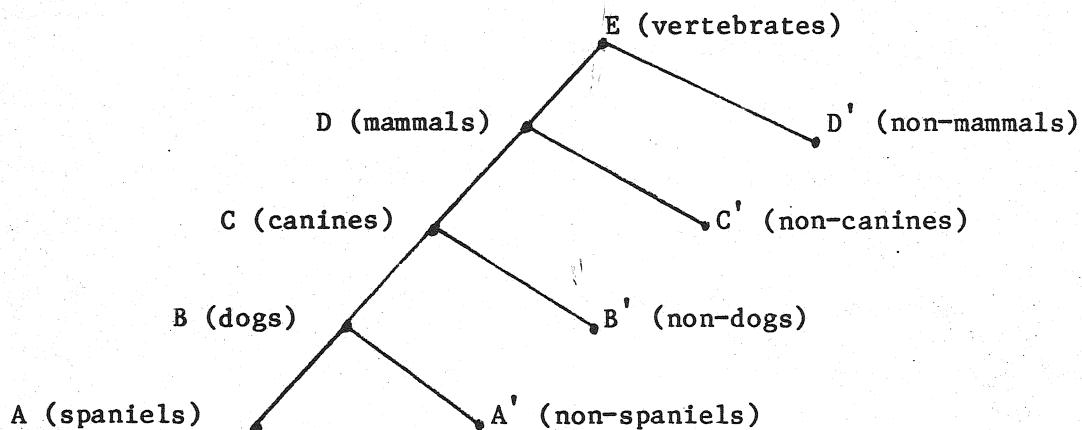


Figure 4. Zoological Hierarchy

The unprimed classes are called primary classes and refer to a singular class (e.g., B=the class of dogs). The primed classes are called secondary classes and refer to all the subclasses within its supraordinate class and excluding the class indicated by its letter unprimed (e.g., B'=all the subclasses within the class of canines, C, excluding the class of dogs, B).

We can mentally pose (by logical addition, +) and unpose (by logical subtraction, -) a class, giving us the elementary operations +D, -D, +C, -C, etc. We may pose a series of operations:

$$(+B) + (+B') + (+C') = (+D)$$

or simply $B + B' + C' = D$ (1)

$$C - B' = B$$
 (2)

Formula (1) tells us that the class of dogs together with the subclasses of non-dogs that are canines, together with the subclasses of non-canines that are mammals gives us the class of mammals ($B+B'=C$ and $C+C'=D$). Formula (2) tells us that posing the class of canines together with unposing the subclasses of non-dogs that are canines gives us the class of dogs, B.

A grouping is defined as a set of elements (here classes) and the logical operation of combining or adding equations of the form $(A+A'=B)$, $(B-A=A')$, $(-D-D'=E)$, etc. A grouping has five properties (obeys five rules): composition (or closure), associativity, general identity, reversibility and special identities. The first four rules are group-derived, the last rule is lattice-derived. The following explication of the groupings of concrete operations follows Flavell (1963) which in turn is based upon Piaget's (1942, 1949) treatment. I make no attempt to give formal proofs of grouping properties.

a. Grouping I - Primary Addition of Classes

1. Composition. The result of combining any element (a class addition equation) with any other by means of logical addition of equations is itself an element (an equation) in the class system (hence closure).

$$(B+B'=C) + (C+C'=D) = (B+B'=D) \quad (3)$$

$$(C-B'=B) + (D-C'=C) = (D-C'-B'=B) \quad (4)$$

Note that in Formula (4), the sum of the two equations is not the sum of the right-hand members of each equation (not $B+C=C$). Note that $(D-C'=C)$, but that C is already "denuded" of B' by $(C-B'=B)$ so that B is the final result of the operations.

2. Associativity. It is clear that the series sum is independent of the order of operations. Formulas (5) and (6) are both equal to Formula (7).

$$[(C-B'=B) + (D-C'=C)] + (E-D'=D) \quad (5)$$

$$(C-B'=B) + [(D-C'=C) + (E-D'=D)] \quad (6)$$

$$(E-D'-C'-B'=B) \quad (7)$$

3. General Identity. One and only one element leaves any other element unchanged whenever both are added to one another. This element is called the general identity element and is defined as the sum of two null classes, $(0+0=0)$.

$$(0+0=0) + (E-D=D') = (E-D=D') \quad (8)$$

4. Reversibility. For all elements in the set, there exists a unique element called its inverse. An element added to its inverse yields the identity element. The inverse of $(C=C'=D)$ is $(-C-C'=-D)$. Thus, if we unpose the class of canines and unpose all the subclasses of non-canines, we are in effect unposing the class of mammals (See Figure 4).

$$(C+C'=D) + (-C-C'=-D) = (0+0=0) \quad (9)$$

5. Special Identities. Note that Figure 4 is a semi-lattice in which a least-upper-bound can be found for any pair of classes. (In a semi-lattice, every two elements have a least-upper-bound but unlike a lattice it is not the case that every two elements have a greatest-lower-bound.) For the general case,

$$Y + X = X \quad (10)$$

where X and Y are class equations and X is a class which subsumes Y.

Because X is the least-upper-bound of Y and X, we can see that any class will play the role of identity elements to all supraordinate classes. This lattice-derived property is the special identity called resorption.

When X is the same class as Y ($X=Y$) we have a special case of Formula (10) where a class is an identity element with respect to itself. This lattice-derived property is the special identity called tautology. Formulae (11) through (13) illustrate these two special identities.

$$(B+B'=C) + (B+B'=C) = (B+B'=C) \quad (11)$$

$$(A+A'=B) + (C+C'=D) = (C+C'=D) \quad (12)$$

$$(-C-C'=-D) + (-D-D'=-E) = (-D-D'=-E) \quad (13)$$

The special identities cause the groupings to have both lattice and group properties and, as Flavell points out, leads to special problems. These problems derive from the many exceptions to the general grouping properties and thus require special rules and conventions (see Flavell, 1963, 176). For example, in all the groupings, the special identities restrict the generality of the associativity law. In Grouping I,

$$[(A+A'=B) + (A+A'=B)] + (-A-A'=B) \neq (A+A'=B) + [(A+A'=B) + (-A-A'=-B)]$$

because the left side of the inequality sums to $(0+0=0)$ while the right side sums to $(A+A'=B)$. Piaget's special rules takes care of these kinds of problems but their inclusion in the theoretical formulations attest to the lack of simplicity and elegance of the mathematics.

b. Grouping II - Secondary Addition of Classes (Vicariences)

Although the elements of groupings are class and relation equations rather than singular classes and relations, it is possible and much more convenient to treat singular classes and relations as the elements of groupings provided that Piaget's special rules are utilized. The remaining groupings will follow this procedure.

Recall that secondary classes refer to all complementary classes under the immediate supraordinate class. Thus, B' refers to such canine classes as wolves, dingos, etc. We can then establish within B' a class $B_2 = \text{wolves}$. Then $B_2 + B_2' = B$ because $B_2' = \text{all classes of non-wolves canines}$. This is the complement of B_2 under the class C . Thus, it is possible to create a series that runs parallel to the initial one, Figure 4, and that rejoins it at the next higher rank primary class. Thus, $B_3 + B_3' = C$, $A_1 + A_1' = B$, etc. These equations are called complementary substitutions or vicariences. The important rule here is that if we are given classes X_1 , X_j and X_1' and X_j' , we can always substitute X_1 for X_j providing we substitute X_1' for X_j' in the same equation.

For example, $B + B'$ may always be substituted for $B_2 + B_2'$. Note that the non-wolves include the dogs and that the non-dogs includes the wolves. Thus, wolves (B_2) plus non-wolves (B_2') sum to canines (C) and that dogs (B) plus non-dogs (B') sum to canines (C).

1. Composition. The sum of two or more vicariances yields a vicariance.

$$(B+B'=B_1+B_1')+(C+C'=C_2+C_2')=(B+B'+C'=C_2+C_2') \quad (14)$$

2. Associativity. This property holds provided that special rules are followed.

3. General Identity. $(0+0=0)$ is unique.

4. Reversibility. The inverse of a posed vicariance is the vicariance unposed. The sum of a vicariance and its inverse is the general identity.

5. Special Identities.

Tautology holds W

$$A+A=A \quad (15)$$

$$A_1+A_1=A_1 \quad (16)$$

$$B_2'+B_2'=B_2' \quad (17)$$

Resorption holds.

$$B+C=C \quad (18)$$

$$B_2'+C=C \quad (19)$$

$$B_2'+C=C \quad (20)$$

$$B+B_2'=B_2' \quad (21)$$

$$A+B_3'=B_3' \quad (22)$$

Formula (21) holds because B is a Subclass of B_2' and Formula (22) holds because B_3' subsumes B which subsumes A.

c. Grouping III - Bi-univocal (one-to-one) Multiplication of Classes

It is possible for classes to be multiplied and divided as well as added and subtracted. We may take the class of mammals, M_1 , and partition it into subclasses, vertebrates (A_1) and invertebrates (B_1). Similarly, we can take the same class of mammals (call it now

M_2) and partition it according to whether the mammals are flying (A_2) or non-flying (B_2). We may multiply one partition by another resulting in the logical product or intersection (greatest-lower-bound of the semi-lattice) of the two. We obtain the largest class which the two partitions comprise in common. See Figure 5.

| | | | |
|-------|-------|-----------|-----------|
| M_1 | M_2 | A_2 | B_2 |
| | A_1 | $A_1 A_2$ | $A_1 B_2$ |
| | B_1 | $B_1 A_2$ | $B_1 B_2$ |

Figure 5. Bi-Univocal Multiplication Of Two Classes Or Series (M_1, M_2)

We establish a one-to-one correspondence between each partition in two or more series. In Figure 5 we obtain a 2×2 matrix where $A_1 A_2$ indicates the subclass of mammals that are vertebrates (A_1) and fly (A_2), and so on. Thus,

$$A_1 A_2 + A_1 B_2 + B_1 A_2 + B_1 B_2 = M_1 \times M_2 \quad (23)$$

A Bi-univocal multiplication may contain more than two series, $M_1 \times M_2 \times M_3$ (where M_3 has three partitions), thus giving us 12 subclasses.

1. Composition. The multiplication of two classes (partitions) gives a class (eg., $A_2 \times B_1 = A_2 B_1$) and the multiplication of two series yields a set of classes (see Formula (23)).

2. Associativity. This property holds. For instance, $(M_1 \times M_2) \times M_3 = M_1 \times (M_2 \times M_3)$.

4. Reversibility. The inverse operation is class division (the abstraction or dissociation of one class from a class product). In our example, $A_1 B_2 \div B_2 = A_1$. If mammals that are non-flying are dissociated from the class of non-flying vertebrates, we end up with the larger class of vertebrates. Class multiplication generates classes smaller in extension whereas, class division generates classes larger in extension.

3. General Identity. $A \div A = Z$. Clearly, Z, cannot be the null class. Z is defined as the largest, most general class possible relevant to the class series we are dealing with. Z is the hypothetical class that contains all the others. For example, if $D_x = \text{animals}$, then $D_x \div D_x = Z$, means that we have removed the class-defining limitation "animalness," leaving the most general possible class relevant to "animalness-nonanimalness." Z might then be appropriately called "the class of beings defined by no specific delimiting qualities (Flavell, 1963, 179). Note that the product of Z with any other class leaves us with that other class (e.g., $A_1 \times Z = A_1$).

5. Special Identities W

Tautology clearly holds:

$$B_1 \times B_1 = B_1 \quad (24)$$

Absorption (rather than resorption which is a property of class addition) holds, because a supraordinate class is always partitioned into its subordinate class:

$$M_1 \times A_1 = A_1 \quad (25)$$

d. Grouping IV - Co-univocal (one-to-many) Multiplication of Classes

Here, one member (partition) of a series is multiplied (set in correspondence with) each of one or more additional series. See Figure 6.

| | K_2 | K_1 | A_1 (sons of x) | B_1 (grandson of x) | C_1 (great-grandson of x) |
|----------------------|-------|-------|-------------------|-----------------------|-----------------------------|
| A_2 (brothers) | | | $A_1 A_2$ | $B_1 A_2$ | $C_1 A_2$ |
| A_2' (1st cousins) | | | — | $A_2' B_1$ | $C_1 A_2'$ |
| B_2' (2nd cousins) | | | — | — | $C_1 B_2'$ |

Figure 6. Co-univocal multiplication of classes

Multiplying K_1 and K_2 we obtain

$$K_1 K_2 = A_1 A_2 + B_1 (A_2 + A_2') + C_1 (A_2 + A_2' + B_2') =$$

$$A_1 A_2 + B_1 A_2 + B_1 A_2' + C_1 A_2 + C_1 A_2' + C_1 B_2 \quad (26)$$

The matrix generated is a triangular one ("—" indicates empty subclasses) and since the empty class is included in the elements for this Grouping (as with all groupings) the same grouping properties hold here as they did for Grouping III.

Groupings V through VIII involve operations that are performed upon relations whose compositions are transitive rather than on logical classes. Asymmetrical relations denote ordered differences between terms, e.g., $5 < 7 \neq 7 < 5$.

e. Grouping V - Addition of Asymmetrical Relations

This grouping constitutes the logical addition (and subtraction) of asymmetrical relations (ordered differences within a series of relations). If the set $\{U, V, W, X, Y\}$ of objects or classes and are linked by the transitive asymmetrical relation, $----->$, then we can use Figure 7 to illustrate this grouping.

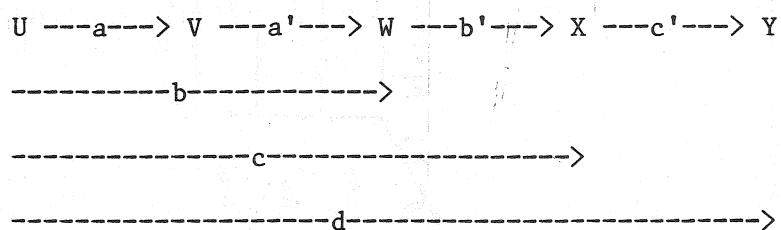


Figure 7. Asymmetrical Transitive Relations

The small letters are the ordered differences indicated by the arrow and satisfy the transitive criterion. Thus,

$$U \rightarrow a \rightarrow V + V \rightarrow a' \rightarrow W = U \rightarrow a + a' \rightarrow W = U \rightarrow b \rightarrow W \quad (27)$$

or more simply: $a + a' = b$.

1. Composition. From the example we can see that this property holds.

2. Associativity. This property clearly holds.

5. Special Identities.

$$\text{Tautology holds:} \quad a + a = a \quad (28)$$

$$\text{Absorption holds:} \quad a + b = b \quad (29)$$

4. Reversibility. The inverse of an ordered difference relation, for instance $W \rightarrow b' \rightarrow X$ is its reciprical, $X \leftarrow b' \leftarrow W$. Thus,

$$(W \rightarrow b' \rightarrow X) + (X \leftarrow b' \leftarrow W) = W \rightarrow o \rightarrow W \quad \text{or} \quad (W = W). \quad (30)$$

3. General Identity. This an equivalence relation, a relation of no difference, o or =. Thus,

$$\rightarrow b' \rightarrow + \leftarrow b' \leftarrow = \rightarrow o \rightarrow \quad (31)$$

These formulae,

$$\rightarrow b \rightarrow + \leftarrow a' \leftarrow = \rightarrow a \rightarrow \quad (32)$$

$$\rightarrow d \rightarrow + \leftarrow b'c' \leftarrow = \rightarrow b \rightarrow \quad (33)$$

are general examples of Figure 7. For instance, if $X \rightarrow c' \rightarrow Y$ means that X is smaller than Y, then $Y \leftarrow c' \leftarrow X$ means that Y is larger than X.

It is important to note that the reversibility property takes two different forms at this stage of intellectual development: inversion for classes and reciprocity for relations.

f. Grouping VI - Addition of Symmetrical Relations

This grouping encompasses the additive compositions of different kinds of symmetrical relations (e.g., transitive, intransitive, reflexive, irreflexive, etc.). Flavell illustrates the workings of this grouping with a genealogical hierarchy in which x, y and z are male members.

Figure 8 shows the relationships that are established.

| Relation Symbol | Relation Meaning | Example(s) |
|-----------------------------------|-------------------------------|--|
| $\langle \text{---o---} \rangle$ | identity | $x \langle \text{---o---} \rangle x$ (or $x=x$) |
| $\langle \text{---a---} \rangle$ | "brother of" | $x \langle \text{---a---} \rangle y$ |
| $\langle \text{---a'---} \rangle$ | "first cousin to" | $x \langle \text{---a'---} \rangle z$ |
| $\langle \text{---b---} \rangle$ | "has the same grandfather as" | $x \langle \text{---b---} \rangle y, x \langle \text{---b---} \rangle z$ |

Figure 8. Genealogical Symmetrical Relations

The difference relations are non-ordered since they are symmetrical. Thus, $\langle \text{---ā---} \rangle$ ("is not the brother of") can be constructed.

1. Composition. Flavell notes that there are many formal rules that Piaget (1949, 154-7) employs for additive composition. Three examples (from Flavell, 182-3) will suffice.

Example 1. $(x \langle \text{---a---} \rangle y) + (y \langle \text{---b---} \rangle z) = x \langle \text{---b---} \rangle z$ (4)

Here, x and y are brothers and x and z have the same grandfather.

Example 2. $(x \langle \text{---a---} \rangle y) + (y \langle \text{---b---} \rangle z) = x \langle \text{---b---} \rangle z$ (35)

Here, if one of the brothers, y , does not have the same grandfather as z , then brother x does not either.

2. Associativity. Clearly associativity holds.

Example 3. $[(r \langle \text{---a---} \rangle x) + (x \langle \text{---a'---} \rangle y)] + (y \langle \text{---a'---} \rangle z) =$
 $(r \langle \text{---a---} \rangle x) + [(x \langle \text{---a'---} \rangle y) + (y \langle \text{---a'---} \rangle z)] =$
 $r \langle \text{---b---} \rangle z$ (36)

4. Reversibility. To obtain the reciprocal operation of any relation, just permute the terms. Thus, the reciprocal of $y \langle \text{---a---} \rangle x$ is $x \langle \text{---a---} \rangle y$.

3. General Identity. This is $x \langle \text{---o---} \rangle x$, or $x=x$. Thus,

$$(y \langle \text{---o---} \rangle x) + (x \langle \text{---b---} \rangle y) = x \langle \text{---o---} \rangle x \quad (37)$$

5. Special Identities.

Tautology holds: $(x \langle \text{---a---} \rangle y) + (x \langle \text{---a---} \rangle y) + x \langle \text{---a---} \rangle y$ (38)

Resorption holds: $(x \langle \text{---a---} \rangle y) + (x \langle \text{---b---} \rangle y) = x \langle \text{---b---} \rangle y$ (39)

g. Grouping VII - Bi-univocal (one-to-one) Multiplication of Relations

This grouping illustrates the one-to-one multiplication of two or more series of asymmetrical relations. The following example demonstrates the nature of this grouping.

Figure 9 represents a set of glass tubes filled with lead and cotton whose sizes and weights vary independently. The letters represent weight (i.e., C is heavier than B which is heavier than A, etc.) and the subscripts represent volume (i.e., 3 is more voluminous than 2 which is larger than 1, etc.). Notice that the tubes are arranged horizontally by increasing size and vertically by increasing weight. The horizontal arrows represent volume differences and the vertical arrows weight differences. Note that all the items in one row have the same volume and all the items in one column have the same weight.

1. Composition. We can multiply a weight relation by a volume relation and obtain a product that is one of weight and volume simultaneously.

Example 1.

$$(A_1 \text{---} c_1 \text{---} A_4) \times (A_4 \downarrow a_2 \text{ } B_4) = (A_1 \text{---} c_1 \text{---} \downarrow a_2 \text{ } B_4) \quad (40)$$

Thus, if A_1 is smaller than A_4 at equal weight and if A_4 is lighter than B_4 at equal volume, then it follows that A_1 is both smaller and lighter than B_4 by the amount $\text{---}c_1\text{---}\downarrow a_2$. (Because groupings involve only intensive quantification, it is not possible to give exact numeric quantities here.)



$$\begin{aligned} \text{Example 2. } (B_1 \text{---} b_1 \text{---} \downarrow a_2' C_3) \times (C_3 \text{---} b_1' \text{---} \downarrow b_2' D_4) = \\ (B_1 \text{---} c_1 \text{---} \downarrow a_2' + b_2' D_4) \end{aligned} \quad (41)$$

$$\begin{aligned} \text{Example 3. } (B_1 \text{---} b_1 \text{---} \downarrow a_2' C_3) \times (C_3 \text{<---} a_1' \text{---} \downarrow a_2' B_2) = \\ (B_1 \text{---} a_1 \text{---} \downarrow B_2) = (B_1 \text{---} a_1 \text{---} \rightarrow B_2) \end{aligned} \quad (42)$$

where <--- means "smaller than" and --- means "heavier than."

2. Associativity. This property obviously holds.

3. Reversibility. The reciprocal operation is logical division, which is analogous to Groupings III and IV.

4. General Identity. This is the null difference for both weight and volume.

$$\begin{aligned} \text{Example 4. } (A_1 \text{---} a_1 \text{---} \downarrow a_2 B_2) (A_1 \text{---} a_1 \text{---} \downarrow a_2 B_2) = \\ (A_1 \text{---} a_1 \text{---} \downarrow a_2 B_2) \times (A_1 \text{<---} a_1 \text{---} \uparrow a_2 B_2) = \\ (A_1 \text{---} o \text{---} \downarrow o A_1) \end{aligned} \quad (43)$$

5. Special Identities.

Tautology holds:

$$\begin{aligned} \text{Example 5. } (A_1 \text{---} a_1 \text{---} \downarrow a_2 B_2) \times (A_1 \text{---} a_1 \text{---} \downarrow a_2 B_2) = \\ (A_1 \text{---} a_1 \text{---} \downarrow a_2 B_2) \end{aligned} \quad (44)$$

Resorption holds:

$$\begin{aligned} \text{Example 6. } (A_1 \text{---} a_1 \text{---} \downarrow a_2 B_2) \times (A_1 \text{---} b_1 \text{---} \downarrow b_2 C_3) = \\ (A_1 \text{---} a_1 \text{---} \downarrow a_2 B_2) \end{aligned} \quad (45)$$

h. Grouping VIII - Co-univocal (one-to-many) Multiplication of Relations

This grouping encompasses the multiplication of symmetrical and asymmetrical relations. These relations define classes in hierarchies. The grouping is illustrated with Flavell's (186) examples. Figure 10 defines the relations in a family tree hierarchy. A, B and C are persons in the tree.

| | Relation Symbol | Meaning |
|----------------------|---------------------------------|--|
| symmetric relations | $\langle \text{--o--} \rangle$ | is the same person as |
| | $\langle \text{--o'--} \rangle$ | is the brother of |
| | $\langle \text{--a--} \rangle$ | is the son of the same father as |
| | $\langle \text{--a'--} \rangle$ | is the first cousin to |
| | $\langle \text{--b'--} \rangle$ | is the grandson to the same grandfather as |
| asymmetric relations | $\downarrow a$ | is the father of |
| | $\uparrow a$ | is the son of |
| | $\downarrow b$ | is the grandfather of |
| | $\uparrow b$ | is the grandson of |

Figure 10. Co-Univocal Multiplication Of Relations

1. Composition. We find that composition here is "formally" analogous to composition in Grouping VII. Multiplication of an asymmetrical relationship by a symmetrical relation gives us a symmetrical-asymmetrical product.

$$\text{Example 1. } (A \downarrow a \ B) \times (B \langle \text{--a'--} \rangle C) = (A \downarrow a \ \langle \text{--a'--} \rangle C) \quad (46)$$

Result: A is the father of the first cousin of C, and thus the uncle of C.

Multiplication of two or more symmetrical-asymmetrical products yields a symmetrical-asymmetrical product.

$$\text{Example 2. } (A \langle \text{--} a' \text{--} \rangle \downarrow b \ B) \times (B \langle \text{--} a' \text{--} \rangle \downarrow a \ C) = (A \langle \text{--} a' \text{--} \rangle \downarrow c \ C) \quad (47)$$

Here, A is the first cousin to the grandfather of B. B is the brother of the father of C and thus is C's uncle. Then A is the first cousin to the great-grandfather of C.

Flavell notes that Piaget's (1942, 182-195 ;1949, 164-166) rules for composition in the Grouping are rather complex.

2. Associativity. This property holds for composition among three or more products of the type $(A \langle \text{--} \text{--} \rangle \downarrow B)$.

3. Reversibility. Reciprocity is the inverse operation which is logical division.

4. General Identity. Dividing a product by itself is the general identity.

$$\text{Example 3. } (A \langle \text{--} a' \text{--} \rangle \downarrow a \ B) \div (A \langle \text{--} a' \text{--} \rangle \downarrow a \ B) = (A \langle \text{--} o \text{--} \rangle \downarrow o \ A) \quad (48)$$

5. Special Identity.

Tautology holds:

$$\text{Example 4. } (A \langle \text{--} a' \text{--} \rangle \downarrow b \ B) \times (A \langle \text{--} a' \text{--} \rangle \downarrow b \ B) = (A \langle \text{--} a' \text{--} \rangle \downarrow b \ B) \quad (49)$$

Resorption holds:

$$\text{Example 5. } (A \langle \text{--} a \text{--} \rangle \downarrow b \ B) \times (A \langle \text{--} a \text{--} \rangle \downarrow a \ C) = (A \langle \text{--} a \text{--} \rangle \downarrow a \ C) \quad (50)$$

There exists a ninth grouping (of equalities) which occurs as a special case of all of the other eight groupings (Piaget, 1942, 33-34). This grouping involves the addition of symmetrical relations of equality.

i. Grouping IX - The Grouping of Equalities

1. Composition. The form of this property is

$$(A=B)+(B=C)=(A=C) \quad (51)$$

2. Associativity. This property clearly holds.

3. Reversibility. The inverse of an operation, $(A=B)$ is $(B=A)$.

4. General Identity. $(A=A)$ (52)

5. Special Properties.

$$\text{Tautology holds: } (A=C)+(A=C)=(A=C) \quad (53)$$

$$\text{Resorption holds: } (A=C)+(D=E)=(D=E) \quad (54)$$

This brief sketch of the logical groupings of concrete operational thought help put the various activities of the child into a more rigorous perspective. While, admittedly, the groupings are not models in the strict sense (they do not precisely describe the child's behavior), they do supply a conceptual framework within which experiments can be carried out and results interpreted. According to Flavell, Piaget views these logical groupings in three distinct lights.

First he views them as a precise and parsimonious structural characterization of "ideal" cognition in the realm of intensive logical operations of classes and relations. Second, they constitute a general framework for interpreting certain global and elusive, but nonetheless important, qualities of concrete operations in contrast to preoperational thought. And finally, they serve as a framework for investigating or "diagnosing for" more specific intellectual attainments in this area. (Flavell, 190)

Numerous empirical studies have shown that concrete operational children do indeed behave to grouping specifications. These children are systematic in their cognitive behavior unlike children in the pre-operational stage of development. It appears that the older children's cognitive behavior derives from a coordinated system of schemes. Another hallmark of concrete operations is that the child's cognitions appear to

have achieved operational reversibility. Reversibility is the key to the system of cognition at this and the next stage of cognitive development. We briefly examine several empirical examples illustrating some of the groupings.

i. Grouping I - Illustrative Example

The concrete operational child is better at composing and decomposing hierarchical classes than the pre-operational child. The older child is more inclined to accurately combine elementary classes into supraordinate classes ($B+B'=C$) as well as decomposing higher-order classes into their subordinate classes. The older child better understands the relation between subclasses and their supraordinate class. $A+A' \rightarrow B$ means that A and A' are seen not only as individual classes but also simultaneously as members of B. The child is able to think of wholes and parts together.

Thus, the older child is able to answer the wooden beads problem (see Item 5) correctly. He knows that B=wooden beads, A=brown beads and A'=white beads. See Inhelder and Piaget (1964). The structure looks like Figure 11.

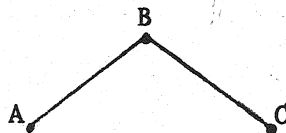


Figure 11. Wooden Bead Experiment

j. Grouping II - Illustrative Example

Piaget has shown that the concrete operational child can classify a collection of objects in several different ways indicating vicariance equations. Thus, C (the total collection of objects) $= B + B' = B_1 + B_1' = B_2 + B_2' = \text{etc.}$ See Inhelder and Piaget (1964). All the different expressions with B 's indicate different classifications within C .

[F]or example "(the French + the non-French) = (the Chinese + the non-Chinese) = (all men)." (Piaget, 1957)

k. Grouping III - Illustrative Example

If a concrete operational child is presented with a horizontal row of pictures of different colored leaves which meets a vertical row of pictures of green colored objects of different kinds, she is able to determine what picture should be placed at the intersection of the class of leaves and the class of green objects (a green leaf). See Inhelder and Piaget (1964).

l. Grouping V - Illustrative Example

Seriation is the key operation of this grouping. Here the child is given three or more objects of differing weight (and the same volume) and is asked to compare only two objects at a time and seriate them all by weight. The pre-operational child is often satisfied to create the correct series, $A < B < C$, by establishing $A < B$ and $A < C$. He creates an incorrect series as often as he produces the correct one. This child is unsure of himself and feels he needs empirical justification to verify $A < C$ knowing $A < B$ and $B < C$. See Inhelder and Piaget (1949).

The concrete operational child, on the other hand, is able to easily deduce $A < C$ from $A < B$ and $B < C$ because she is able to view each element in an asymmetrical series in terms of direct ($<$) and inverse ($>$)

relational operations. The ability to grasp this reversibility also aids the child in the length-transitivity task (see Item 3).

m. Grouping VI - Illustrative Example

The pre-operational child is unable to grasp the symmetry property of symmetrical relations (if $A \rightarrow B$, then the inverse operation holds, $B \leftarrow A$). These children, unlike the concrete operational child, will affirm that while X is their brother, X does not have a brother. See Piaget (1928.)

n. Grouping VII - Illustrative Example

The concrete operational child, when presented with 49 pictures of leaves which can be ordered by size (A to G) and by shade of color (1 to 7), will arrange them in double entry table (Figure 12).

| | | | | | |
|-------------|----|------------------|----|-------|--|
| | | Larger -----> | | | |
| | A1 | A2 | A3 | | |
| | B1 | B2 | B3 | | |
| | C1 | C2 | C3 | | |
| | . | . | . | | |
| | . | . | . | | |
| | . | . | . | | |
| Darker ↓ | | | | | |

Figure 12. Double Entry Table Of Leaf Size And Color

Almost all of Piaget's conservation studies involve the manipulation of relations like this one:

$$(A \rightarrow B) \times (A \uparrow B) = (A \rightarrow \uparrow B). \quad (55)$$

Here, the equality of two objects along various dimensions (length, quantity, area, volume, etc.) is conserved in the face of a

transformation of one. For instance, see Item 4. If A and A' are equal quantities of beads in glass jars and A' is poured into a wider beaker, (A'---->B), then the concrete operational child will conserve quantity because he will be able to multiply the relations "lower than" and "wider than" to see that the initial quantity is conserved (i.e., still equal). See Inhelder and Piaget (1964).

According to Flavell, no experimental evidence exists to verify Groupings IV and VIII. Piaget apparently invented these two groupings because "they describe logically possible structures, not empirically discovered (as yet, at least) logical structures." (Flavell, 1963, 189)

o. Grouping of Equalities - Illustrative Example

Given three objects of equal weight but differing in size and color, the preoperational child, unlike the concrete operational child, will doubt that $A=C$ when she empirically establishes by a balance that $A=B$ and $B=C$.

The closure of the concrete operational structure (represented by the nine groupings) brings about adult logical necessity and self-evidence to a limited but important degree (i.e., reversibility and hence conservation). However, full maturation must wait for the next stage of intellectual development.

3. Concrete Operational Limitations

While this stage of intellectual development is a significant advance over the previous one, it does have its limitations. The pre-adolescent is limited to organizing only immediately given and perceived data.

The concrete operational child remains attached to empirical reality. What she is unable to do is to generate all of the possibilities inherent in a situation and then try to discover which, if any, of these possibilities occur in the actual data of the situation. It is in this paradigm that the real becomes a special case of the possible and is not achieved until the next developmental stage.

Finally, the child at this stage is unable to exhibit a structural organization that is interwoven into an integrated system. He possesses two types of reversible operations, negation (or inversion) and reciprocity, which pertain to class and relational groupings, respectively. This child has not yet developed a system which can coordinate the two reversibilities and thus allow him to solve "multivariable" problems.

4. Transductive Reasoning

With the onset of reversibility of thought comes the extinguishing of transductive reasoning. This is, at least, the conventional view. Transductive reasoning is neither true inductive nor true deductive reasoning, but proceeds from particular to particular. Flavell puts it very well:

Centering on one salient element of an event, the child proceeds irreversibly to draw as conclusion from it some other, perceptually compelling happening. Piaget makes the important point that the factual correctness of the child's conclusion ... is by itself no guarantee that the mechanism for arriving at it was logical rather than transductive. (Flavell, 1963, 160)

The child who reasons transductively tends to juxtapose elements, thus making associative "and" connections. There is nothing that can be termed "implicative" or "causal" between elements or events. As Piaget puts it:

[J]uxtaposition is after all the sign of the complete absence of necessity from the thought of the child. The child knows nothing either of physical necessity (the fact that nature obeys laws) nor of logical necessity (the fact that such a proposition necessarily involves such another). For him everything is connected with everything else, which come to exactly the same thing as nothing is connected to anything else. (Piaget, 1928, 60)

Thus, transduction is inference from particular to particular in the absence of any general law and logical rigor. General laws and logical rigor are absent because the possibility of logical reversibility is absent. Piaget describes transductive reasoning as an "elementary mental experiment" and claims that it is not yet

... a process of necessary reasoning; there is nothing necessary about the observation of facts so long as the elements of reality so observed are not dissociated to the extent of supplying the material for the construction of a simpler and completely reversible reality. (ibid., 190)

"Pure transduction" extends to age 7-8 and is thought to be extinguished by the onset of reversible operations. In the concrete operational stage, 7-8 to 11-12 years, some mental experiments become reversible and the beginnings of logical or theoretical necessity appears. The child now wants to connect two associated phenomena by a necessary relation rather than by simply recalling their common history.

Complete (or full) reversibility of thought, however, does not occur until the formal operations stage of cognitive development. Thus, transductive reasoning is not fully extinguished and replaced by logical necessity until this stage is reached. Traces of, or regression to, transductive reasoning seems to occur late into formal operations (Seidman, 1980). This is an example of what Piaget would term, "vertical decalage."

Transductive reasoning has its roots in the egocentrism of the child and it is not until the child begins to lose this egocentrism that transductive logic begins to give way to logical necessity. Piaget's (1926; 1928) most detailed account of these roots and transductive reasoning appears in his "Studies in Child Logic": The Language and Thought of the Child (Volume I) and Judgement and Reasoning in the Child (Volume II), respectively. In the former work, Piaget shows that language reflecting the implicative relation is rarely used by younger children to mean logical implication. These works precede Piaget's formalization of the developmental stages in mathematical terms and thus there is no formal model of transduction.

It is well worth quoting a succinct and definitive statement on transductive logic and its relation to the development of logical necessity.

In short, transduction is a combination of elementary relations, but without reciprocity of these relations amongst each other, and consequently without the element of necessity that would lead to generalization. As soon, on the other hand, as relations become completely reciprocal, the fertility of relational multiplication knows no bounds, and generalization becomes possible. Nay, more, this reciprocity is what explains the reversibility of all deductions and consequently the character of strictness and necessity that is peculiar to the reasoning process. (ibid., 197-8)

D. Psycho-logic Model II - The Structure of Formal Operations

1. Hypothetico-Deductive Reasoning

The hallmark of this period of intellectual development is the reversal of thinking between what is real and which is possible, in the child's approach to problems.

Unlike the concrete operational child, the adolescent begins his consideration of the problem at hand by trying to envisage all possible relations which could hold true in the data and then attempts, through a combination of experimentation and logical analysis to find out which of these possible relations in fact hold true. Reality is thus conceived as a special subset within the totality of things which the data would admit as hypotheses; it is seen as the "is" portion of a "might be" totality, the portion it is the subject's job to discover. (Flavell, 205-6)

Hypothetico-deductive thought is deduction at maturity; matured from its naive (concrete) state it will be used by the adolescent throughout adult life. This hypothetico-deductive reasoning frees the adolescent from her dependence upon the material world, because now she can generate all possibilities in the form of hypotheses.

[I]nstead of deriving a rudimentary type of theory from the empirical data as is done in concrete inferences, formal thought begins with a theoretical synthesis implying that certain relations are necessary and thus proceeds in the opposite direction. Hence, conclusions are rigorously deduced from premises whose truth status is regarded only as hypothetical at first; only later are they empirically verified. (Inhelder and Piaget, 1958, 251, emphasis mine)

Formal thought (formal operations) entails propositional thought.

The entities manipulated are no longer concrete data but assertions (propositions) which have these real world data as their content. If we can describe the concrete operations of classifying, seriating, corresponding, etc., as first-degree operations, then operations upon the results of first-degree operations may be considered as second-degree operations. The adolescent casts first-degree operations into propositions and operates upon these propositions with such logical operations

as implication, identity, disjunction and conjunction.

Concrete operations are intrapositional since they result in the content of individual propositions. Formal operations are termed interpropositional because they are relations among propositions formed by concrete operations. Thus, empirical reality is subordinated, in this stage of development, to a system of hypothetico-deductive operations (that which is possible).

The reversal of direction between the real and the possible is most clearly illustrated in the adolescent's propensity for subjecting problem variables to a combinatorial analysis. He wants to determine all the possible relations in the problem so that he may subject them to his reality tests to see which ones hold. He thus systematically isolates all individual variables and generates all possible combinations.

The system within which hypothetico-deduction, propositional and combinatorial thought is couched is a structure that is vastly more mature than the preceding one. While inversion and reciprocity refer to two different systems (classes and relations, respectively) in the concrete operational stage, now they are combined in such a way so as to make this distinction unnecessary. The result is the ability of the adolescent to test the causal efficacy of a variable in an experiment. While negation (or inversion) entails the literal removal of a variable from operation, reciprocity entails only its neutralization. Reciprocity holds the variable's effect constant while a second factor is being varied. Consider this illustrative example from Flavell:

[W]here the problem is to study the separate effects of kind of metal and length on the flexibility of a rod, the younger child finds himself at an impasse; he cannot literally negate either variable, i.e., work with a rod not made of some metal and not possessing some length. The older child uses the reciprocal operation with great profit here. He takes two rods of different metals but of the same length (here length is not negated, but neutralized or controlled - not lengths per se but length differences are annulled) in order to study the effect of kind of metal, and two rods of single metal and different lengths to study the effect of length. (ibid., 210)

Unlike concrete operations, formal operations can be described by abstract models that are complete lattices and groups (rather than semi-lattices and groupings), integrated within one total system.

2. Formal Operations Model

We begin the model explication with the interpropositional operations. Since the adolescent is able to conceive of all possibilities and then can observe and experiment to see which of these logically possible entities occur, he can make logical deductions about the causal nature of the system he is experimenting with. These hypothetical possibilities have a lattice structure.

First we must examine the transition from concrete operational groupings of classes and relations to these propositional structures. Suppose A represents the occurrence of some event and A' the nonoccurrence of even A. B indicates that some variable is present and B' indicates that variable B is not present. The concrete operational child can systematically observe and record some limited associations between event A and the variable B. Utilizing Grouping III he might establish, on the basis of the data, that the following products occur: $(AXB)+(AXB')+(A'XB)+(A'XB')$. Thus, A occurs with B present and with B absent. In addition, A does not occur with B present and absent.

For the adolescent these associations have a propositional significance as opposed to a class-product significance. AXB' is a hypothetical assertion that two statements ("A occurs" and "B is present") conform to the data and their truth can be jointly asserted at a given moment in time.

The change from intrapropositional operations to interpropositional operations is best demonstrated by a change in symbolism. See Figure 13. Beth and Piaget (1966, 181 - hereafter also Piaget, 1966) makes it clear that he utilizes propositional symbols and symbolic logic concepts in a purely descriptive manner in order to make structural analysis easier.

| <u>Intrapropositional expression</u> | <u>Interpropositional expressions (elements)</u> | <u>Element Symbol</u> |
|--|--|-----------------------|
| AXB | $p \cdot q$ | (i) |
| AXB' | $p \cdot \bar{q}$ | (ii) |
| $A'XB$ | $\bar{p} \cdot q$ | (iii) |
| $A'XB'$ | $\bar{p} \cdot \bar{q}$ | (iv) |

Figure 13. Intrapropositional And Interpropositional Expressions And Symbols

Note, in Figure 13, that class symbols A and B are replaced by the propositional symbols p and q, respectively. Also, class multiplication (X) and addition (+) are replaced by propositional conjunction (.) and disjunction (v), respectively. For example, $(AXB) + (AXB') + (A'XB) + (A'XB')$ is now expressed as $(i) \vee (ii) \vee (iii) \vee (iv)$. We can now see how the adolescent generates all possible combinations of these associations. Consider the truth-table for element (i) shown in Figure 14. This table can be represented schematically by Figure 15.

| p | q | $p \cdot q$ |
|---|---|-------------|
| T | T | T |
| T | F | F |
| F | T | F |
| F | F | F |

Figure 14. Truth-Table For Element (i)

| | | | |
|-----|-------------|-----------|-----------|
| | | p | \bar{p} |
| (i) | $p \cdot q$ | q | T |
| | | \bar{q} | |

Figure 15. Schematic Representation For Element (i)
Note: blank cells indicate F

Figure 16 shows the remaining representations for the other three elements (they are derived from their truth-tables).

| | | | | | | | |
|------|-------------------------|-----------|-----------|--|-------|-------------------|-----------|
| | | p | \bar{p} | | | p | \bar{p} |
| | | q | | | | q | T |
| (ii) | $p \cdot \bar{q}$ | \bar{q} | T | | (iii) | $\bar{p} \cdot q$ | |
| | | | | | | \bar{q} | |
| | | p | \bar{p} | | | p | \bar{p} |
| | | q | | | | q | |
| (iv) | $\bar{p} \cdot \bar{q}$ | \bar{q} | | | | \bar{q} | T |

Figure 16. Schematic Representations For Elements (ii), (iii), AND (iv)

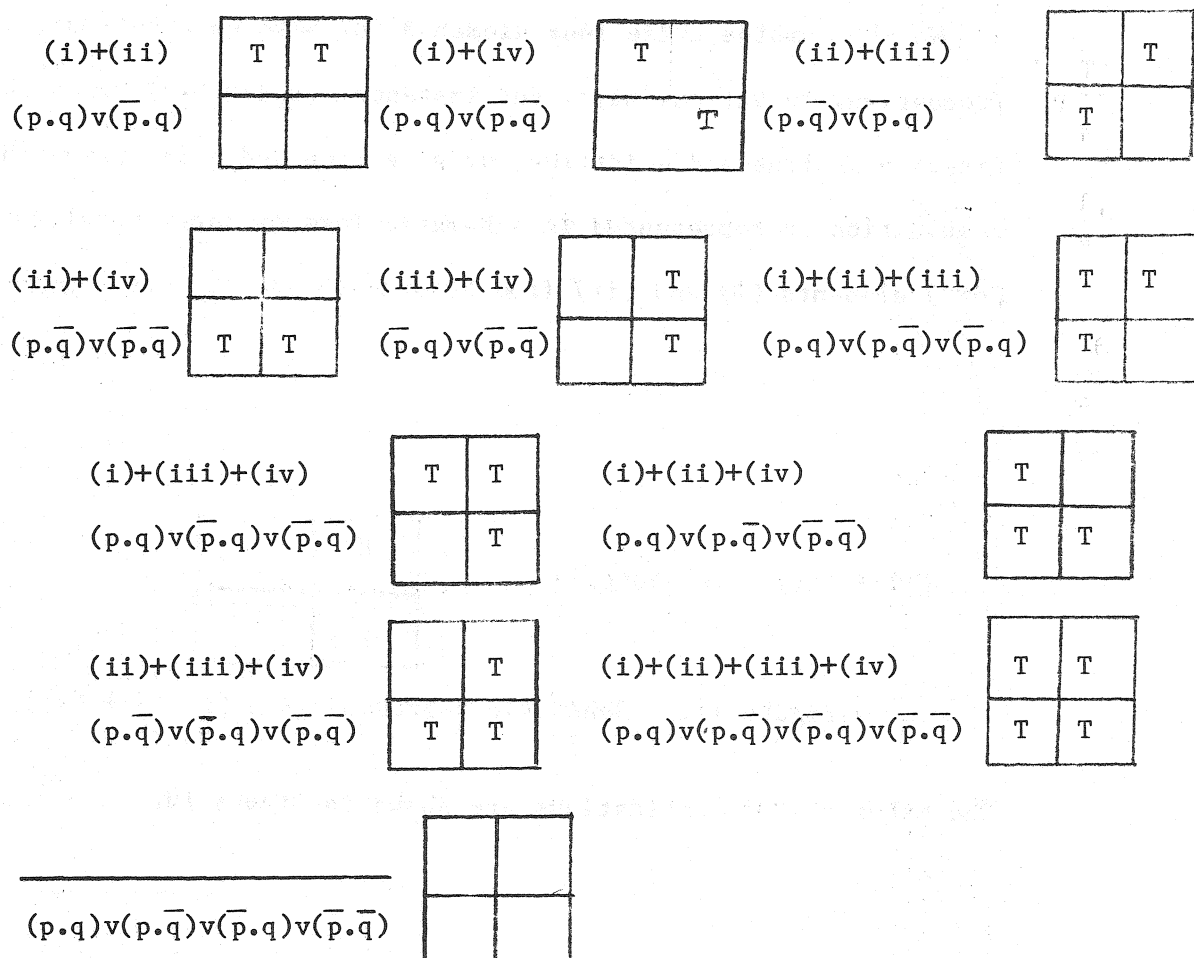
We may combine these four elements (or base combinations) of propositions in various ways. For instance, $(p \cdot q) \vee (p \cdot \bar{q})$ means that either p is true and q is true, or p is true and q is false. This combination is represented in schematic form by superimposing (logical "or") elements (i) and (ii) from Figures 15 and 16. See Figure 17.

$(i) + (ii) \quad (p \cdot q) \vee (p \cdot \bar{q})$

| | | |
|-----------|-----|-----------|
| | p | \bar{p} |
| q | T | |
| \bar{q} | T | |

Figure 17. Schematic Representation Of $(i) + (ii)$

The other eleven combinations are shown in Figure 18.



(Note: This is the symbolic equivalent of the empty schematic)

Figure 18. Remaining Combinations Of Elements
(Note: The proposition labels have been left off the schematics)

Figure 19 gives each combination a combination number.

| <u>Combination #</u> | <u>Combination</u> | <u>Combination #</u> | <u>Combination</u> |
|----------------------|---------------------|----------------------|---------------------|
| 1 | 0 (empty schematic) | 9 | (ii)+(iii) |
| 2 | (i) | 10 | (ii)+(iv) |
| 3 | (ii) | 11 | (iii)+(iv) |
| 4 | (iii) | 12 | (i)+(ii)+(iii) |
| 5 | (iv) | 13 | (i)+(iii)+(iv) |
| 6 | (i)+(ii) | 14 | (i)+(ii)+(iv) |
| 7 | (i)+(iii) | 15 | (ii)+(iii)+(iv) |
| 8 | (i)+(iv) | 16 | (i)+(ii)+(iii)+(iv) |

Figure 19. Combinations And Their Corresponding Numbers

These 16 combinations are called "binary operations" because they derive from combining four unitary values p , \bar{p} , q , and \bar{q} in all possible ways (i.e., the 16 different possible functions derived from p , q combinations). The set of all possible combinations of propositions p and q form a lattice structure. The combination of the lattice elements and the propositional operation, logical "and" (\cdot) gives a unique greatest-lower-bound (glb) for any two elements. The propositional operation, logical "or" (\vee) gives a unique least-upper-bound (lub) for any pair of elements. See Figure 20. This lattice is generated by operations which derive from three concrete operational groupings (I, II and III). See Piaget (1958, 289-93) for a complete explication.

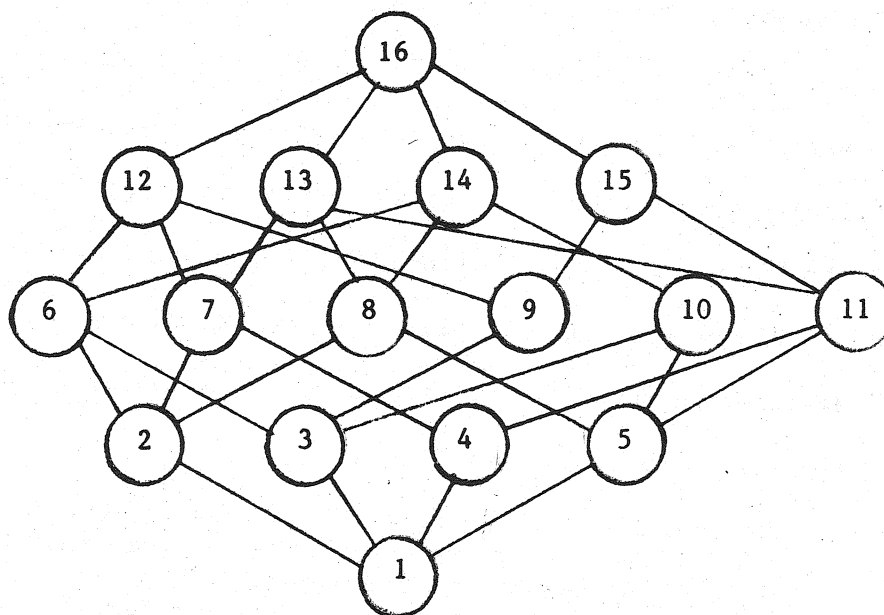


Figure 20. Lattice Structure For All Possible Binary Combinations Of p , q

Piaget expresses the 16 combinations as propositional relations. Consider $p \vee q$, which means "either p or q or both." What are the conditions under which $p \vee q$ is true? Note that $p \vee q$ is compatible with the truth of both p and q , $(p \cdot q)$; the truth of p and falsity of q , $(p \cdot \bar{q})$;

the falsity of p and the truth of q , $(\bar{p}.q)$; any two cases out of the three or all three (Piaget's interpretation of the last two conditions is explained below).

The truth of $p \vee q$ is, however, incompatible with the falsity of both p and q , $(\bar{p}.\bar{q})$. In Piaget's notation:

$$p \vee q = (p.q) \vee (p.\bar{q}) \vee (\bar{p}.q) \quad (56)$$

but $p \vee q / (\bar{p}.\bar{q})$, where "/" stands for incompatibility.

In ordinary propositional logic it would be impossible to assert that two of the above combinations hold simultaneously. For instance, $p \vee q = (p.q) \vee (p.\bar{q})$. If $(p.q)$ is to hold then p and q must be true. But if $(p.\bar{q})$ is to hold then p must be true and q must be false. Clearly, q cannot be both true and false simultaneously.

Piaget notes that the new system of possibilities (Figure 20) resulting from the combinatorial operations on the four elements is no longer a simple classification system but a lattice structure based on the "structured whole" of n -by- n combinations.

Piaget uses the symbols of propositional logic for purely descriptive purposes. He defends his special use of the \vee operation in the following quotation:

For example, the subject will eventually want to know whether two properties x and y are mutually exclusive (from whence $x.\bar{y} + \bar{x}.y$) or whether they are simply disjunctive although they may appear together (from whence $x.y + x.\bar{y} + \bar{x}.y$). When he asks such a question, the subject's reasoning deals not with reality directly but rather with reality as a function of possibility. Here addition (+) is no longer an addition of possibilities, for the real cases cannot always occur simultaneously. This is why the fundamental operation of propositional logic is noted \vee in the sense of "or": thus $x \vee y$ signifies "either $x.y$ is true, or $x.\bar{y}$, or $\bar{x}.y$, or two of these cases out of three, or all three." (Inhelder and Piaget, 1958, 292)

Piaget's logic is not so easily disposed of and we shall return to examine some of its idiosyncrasies. As Ennis (1975) points out, it appears that Piaget means a special sort of "conjunction" when he refers to v . The above Piaget quote will due for now but as we shall later see, it is anything but clear.

Piaget uses the symbol \supset , to stand for material implication (the material conditional). I shall use the symbol \rightarrow , in this text. Thus, "if p then q " means that the relationship could hold in these cases: both p and q are true, $(p.q)$; p is false but q is true, $(\bar{p}.q)$; both p and q are false, $(\bar{p}.\bar{q})$. Note that $p \rightarrow q$ is incompatible with the truth of p and the falsity of q , $(p.\bar{q})$. Thus,

$$p \rightarrow q = (p.q) \vee (\bar{p}.q) \vee (\bar{p}.\bar{q}) \quad (57)$$

but $(p \rightarrow q) / (p.\bar{q})$.

Note that $p \rightarrow q$ is combination number 14 in Figure 20. Combinations 12, 13 and 15 can be replaced with their equivalent implication notations. See Figure 21. Similarly, combinations 6 through 11 can be replaced by equivalent forms. If we look at combination 6 we see that is true regardless of the truth value of q . Thus, combination 6 becomes, p . In combination 9, observe that when p is true q is false and that when p is false q is true. Thus, combination 9 becomes p/q . Figure 20 may be rewritten (without the connecting lines). See Figure 21.

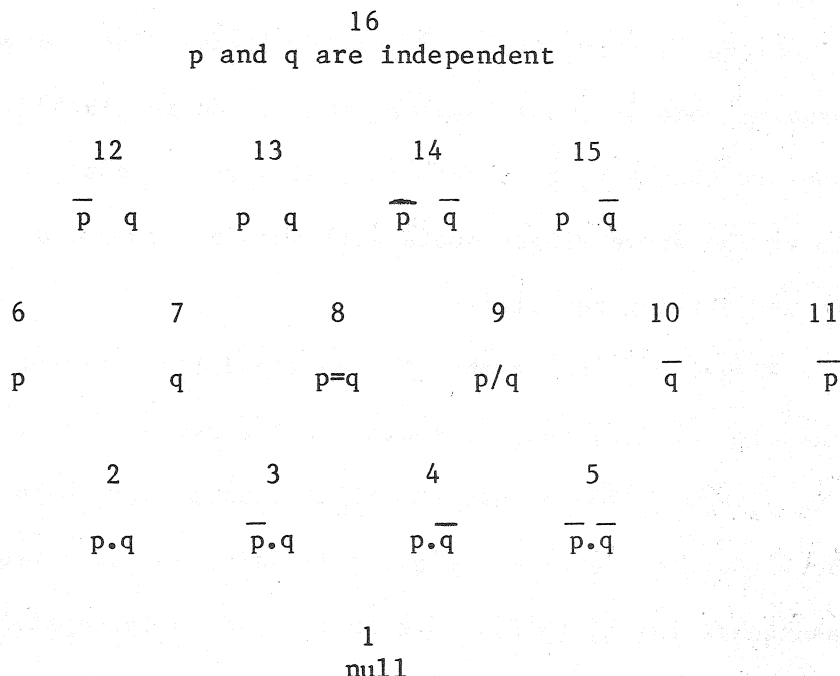


Figure 21. The Sixteen Binary Combinations (Relations)
Expressed As Propositional Relations (from Boyle, 83)

Higher order relations and operations are possible and although Piaget claims that adult thought makes use of them he never explicitly considers them and neither will I.

An example is in order (from Inhelder and Piaget, 1958, Chapter 7).

Item 13. Combinations. The subject is given four identical beakers containing (1) dilute sulphuric acid, (2) water, (3) oxygenated water, (4) sodium potassium and (g) potassium iodide. All liquids are colorless. In addition, two other beakers (i) one containing dilute sulphuric acid and oxygenated water (i.e., (1)+(3)) and (ii) the other containing water are present. Once again, (i) and (ii) are colorless. The experimenter now adds several drops of g to beakers (i) and (ii). The liquid in (i) turns yellow while that in (ii) remains colorless.

It is the subject's task to mix the contents of beakers 1 through 4 in such a manner so as to obtain a yellow color when g is added. He must mix 1 and 3 but not add 4 because sodium thiosulphate cancels the effects of potassium iodide leaving the liquid colorless. Thus, ((1+3+g)+4) reverses the action (i.e., reciprocity).

Concrete operational children are limited to simple multiplicative factoring (Grouping III). They multiply each of the beakers, 1 through 4 by g in an unsystematic trial and error method. The formal operational child, on the other hand, behaves differently. Let p stand for the presence of color, q for beaker 2 (i.e., the water). We can describe the

effect of adding 2 to $(1+3+g)$. The adolescent may note that $p.q$ occurs, i.e., the yellow color occurs when water is added to the mixture $(1+3+g)$. The concrete operational child who is not thinking in terms of the totality of all possibilities might deduce, $p \rightarrow q$. The adolescent on the other hand will not make this deduction on a single observation out of the logical totality. The adolescent, upon further experimentation, will discover: $p.\bar{q}$, $\bar{p}.q$, $\bar{p}.\bar{q}$. He thus obtains the combination: $(p.q) \vee (p.\bar{q}) \vee (\bar{p}.q) \vee (\bar{p}.\bar{q})$. He can then conclude that p and q are independent.

If we now let q stand for beaker 4, we get: $(p.\bar{q}) \vee (\bar{p}.q)$, indicating that color is present in the absence of 4 and that color is absent in the presence of 4. Clearly, p/q .

If, for instance, further experiments establish the presence of $p.q$, $p.\bar{q}$, $\bar{p}.\bar{q}$ and the nonexistence of $\bar{p}.q$ then we would have $(p.q) \vee (p.\bar{q}) \vee (\bar{p}.\bar{q})$ and could conclude, $\bar{p} \rightarrow \bar{q}$.

3. The INRC Group

The interpropositional operations described above have group as well as lattice properties. Certain aspects of cognitive behavior demonstrate a cognitive structure with properties of a four group (group of four transformations, the "Vierergruppe" or "Klein group"). A set of four transformations, I, N, R, and C are the elements of the group under the combination (or multiplication) operation. The transformations are:

1. Identity (I). Nothing in the proposition upon which this transformation is performed is changed.

Example 1. $I(pvq) = pvq$; $I(\bar{p}.q) = \bar{p}.q$; $I(p \rightarrow q) = p \rightarrow q$.

2. Negation (N). The negation of a proposition is simply the proposition that it is incompatible with.

Example 2. $N(p \rightarrow q) = p.\bar{q}$; $N(pvq) = \bar{p}.\bar{q}$; $N(p.q) = \bar{p}v\bar{q}$.

Note that except for \rightarrow , all assertions (propositions) become negations and all negations become assertions. All conjunctions are changed to disjunctions and visa versa.

3. Reciprocal (R). Here assertions (propositions) and negations are merely permuted. All connectors remain unchanged.

Example 3. $R(pvq) = \bar{p}v\bar{q}$; $R(\bar{p}.q) = p.\bar{q}$; $R(p \rightarrow q) = \bar{p}.\bar{q} = q \rightarrow p$

4. Correlative (C). Here conjunctions and disjunctions are permuted but assertions and negations remain unchanged.

Example 4. $C(\bar{p}vq) = \bar{p}.q$; $C(p.q) = pvq$; $C(p \rightarrow q) = \bar{p}.q$ because
 $p \rightarrow q = \bar{p}vq = (p.q) \vee (\bar{p}.q) \vee (\bar{p}.\bar{q})$ and since
 $p \rightarrow q$ and $\bar{p}vq$ are both $\neg p.q$ thus
 $C(p \rightarrow q) = C(\bar{p}vq) = \bar{p}.q$.

The group properties of these transformations are shown below (no formal proofs of these properties are given).

1. Composition. It is easily shown that multiplication of any two or more transformations results in a singular transformation of the four group.

Example 1. $INR(pvq) = C(pvq) = p.q$ (thus, $INR = C$)

2. Associativity. It is easily shown that this property holds.

Example 2. $IN(pvq) = NI(pvq) = \bar{p}.\bar{q}$ (thus, $IN = NI$)

3. General Identity. I is the identity element.

Example 3. $IR = R$; $IC = C$

4. Inverse. Each element is its own inverse.

Example 4. $RR = I$; $CC = I$

Figures 22 and 23 illustrate group of four transformation relationships.

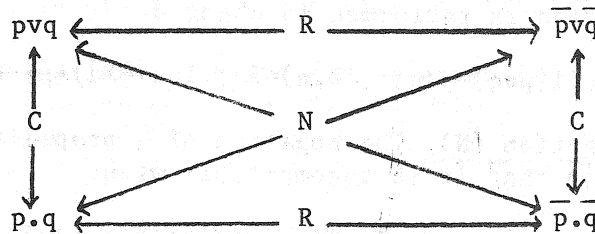


Figure 22. Group Of Four Transformations

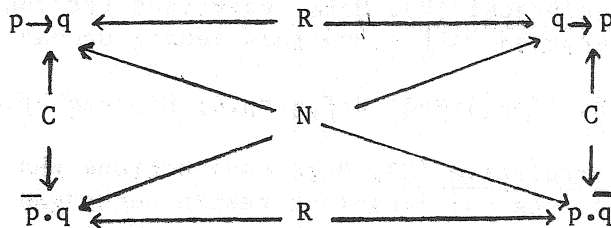


Figure 23. Another Group Of Four Transformations

Thus, in the formal operational stage of development, we see that negation (inversion) and reciprocity are elements of the same system and that this system, the group of four transformations, is closed under composition. Item 14 illustrates how the logical INRC system can be applied to a physical system (from Inhelder and Piaget, 1958).

Item 14 - Balance Arm. Suppose we have a simple two-armed balance and some weights (e.g., 5, 10, 15 and 20 units) that can be inserted in holes spaced regularly along both of the arms. The downward pull on the balance arm is the product of the weight and its distance from the fulcrum point (i.e., the moment of inertia). By asking the subject to explain and predict the outcomes of arranging different combinations of weights and distances we can investigate the growth of understanding of the principles of reasoning.

Let p stand for an increase and \bar{p} a decrease of weight on arm A. Let q stand for an increase and \bar{q} a decrease of distance on A. We say that p' , \bar{p}' , q' , \bar{q}' stand for analogous properties acting on arm B. If the balance is put out of equilibrium (in a downward direction) by an action on arm A we can invert the action by either decreasing the weight (\bar{p}) or by decreasing the distance (\bar{q}). Thus, $N(p.q) = \bar{p}\bar{q}$.

This leads to the problem that if the balance is put out of equilibrium by p and q ($p.q$) it could be put back into equilibrium by either \bar{p} or \bar{q} alone or both p and \bar{q} together. Thus, \bar{p} or \bar{q} acting alone would not be true negation. This problem does not, however, compromise the illustrative value of this Item. See Parsons and Flavell for a full explication of this problem.

Alternatively, we could restore the equilibrium by the weight and distance on arm B, ($p'.q'$). This is reversal by reciprocity. But, increased weight and distance on A is equivalent to decreased weight and distance on arm B. Thus, $p'.q' = \bar{p}.\bar{q}$, and $R(p.q) = \bar{p}'.\bar{q}'$.

We can alter the effect of $p.q$ on arm A by decreasing the weight on arm A, or by decreasing the distance on A, $\bar{p}\bar{q}$. But, the effect of decreasing these weights and distances on A is the same as increasing them on B, $\bar{p}\bar{q} = p'.q'$.

Thus: $I(p.q) = pq$; $R(p.q) = \bar{p}'.\bar{q}'$; $N(p.q) = \bar{p}\bar{q}$; $C(p.q) = pq$.

It should be noted that the adolescent, like his concrete operational counterpart, remains unaware of the formal symbolic logic involved in describing the algebraic structures.

In the formal stage of development, we witness not merely a juxtaposition of inversions and reciprocities but a synthesis, a fusion of both into an integrated whole. As Piaget puts it:

Henceforth every operation will at once be the inverse of another and the reciprocal of a third, which gives four transformations: direct [identity], inverse, reciprocal, and inverse of reciprocal, the latter also being the correlative (or dual) of the first. (Piaget, 1969, 138-9)

Item 15 - Implication. This is an example of what Piaget terms, implication ($p \rightarrow q$). An Adolescent observes an object that keeps stopping and starting and also notices that the stops are accompanied by the lighting of a bulb. His first hypothesis is that the light (p) is the cause or the indication of the stops (q), thus, $p \rightarrow q$. We know that according to Piaget's logic: $p \rightarrow q = (p \cdot q) \vee (\bar{p} \cdot q) \vee (\bar{p} \cdot \bar{q})$ but that $p \rightarrow q / p \cdot \bar{q}$. Thus, to confirm the hypothesis (he has already confirmed $p \cdot q$) the adolescent must discover whether $p \cdot \bar{q}$ occurs. If it doesn't then the hypothesis holds, otherwise, the hypothesis is disconfirmed.

He may also wonder whether the light is caused by the stop, $q \rightarrow p$, the reciprocal of $p \rightarrow q$. He must now discover whether $q \cdot \bar{p}$ holds in order to confirm or disconfirm this hypothesis. Note that $\bar{p} \cdot q$ is the inverse of $p \rightarrow q$ and at the same time the correlative of $p \rightarrow q$. Similarly, $p \cdot \bar{q}$ is the inverse of $q \rightarrow p$ and the correlative of $q \rightarrow p$. In the first case the object stopping each time the light lights is compatible with its sometimes stopping for another reason. And in the second case, if every time there is a stop the bulb lights ($q \rightarrow p$) there can be lights without stops. Note that if $q \rightarrow p$ is the reciprocal of $p \rightarrow q$ then $\bar{p} \cdot q$ is the reciprocal of $p \cdot \bar{q}$. See Piaget (1969, 139).

While there are a number of logical and Piagetian idiosyncratic problems with Item 15, it does serve to illustrate that while the adolescent is not aware explicitly of any logic he nonetheless behaves as if he is capable of manipulating hypotheses according to the INRC group. These transformations of propositional relations then, provide a model for describing the adolescent's problem solving thought processes. Note that the transformations are really third degree operations since their contents are already second degree operations. The child at this stage of development is now fully freed of the necessity of physical action and can truly think and manipulate free from the restrictions of concrete reality.

4. Reflective Abstraction and the Transition from Concrete to Formal Operations

Since we approach stage development from a structural framework we can observe just how the phenomena of reflective abstraction gives rise to the final stage of cognitive structures. This transition is usually described from an equilibrium standpoint (Flavell; Inhelder and Piaget, 1958). Piaget briefly describes it from a structural standpoint (Piaget, 1966) and I shall draw from that description.

The process that characterizes the transition from the concrete to formal operations stage can, according to Piaget, be generalized. To go from one structure to the next one must abstract operational relations from the antecedent structure in order to generalize them in the new one. These "reflected" relationships continue the old operations in a new "plane." Thus, we have continuity and novelty, two necessary ingredients in the transition. Finally, these new operations permit heretofore separated systems to be combined into new structured wholes.

This transition is important for it now allows the substitution of deduction for experience. Deduction, at this high level, now is able to deal with pure hypotheses whereas in the previous level it was "shackled" by concrete objects and sense experience.

Concrete operations form a limited system arising out of the structure of groupings and which function only when objects in the environment are manipulated. The fundamental deductive concepts marked by concrete operations are the concepts associated with conservation.

From the functional viewpoint, these same concrete operations exhibit a general limited character which is very instructive: they only function in the presence of objects, when the latter are manipulated, or supported by representations, but only insofar as the latter directly continue the possible manipulations and they become useless when the objects are replaced by simple hypotheses stated verbally. (Piaget, 1966, 239)

Thus, the manipulation of objects is a necessary condition for these first conservatory deductions to occur. The deductions are based upon the transitivity of overlapping classes or on the transitivity of symmetrical or ordered asymmetrical relations.

The formal stage is marked by the freeing of content from reasoning. The formal operational child, according to Piaget, is now able to "reason deductively about simple hypotheses stated verbally" (ibid., 240). The appearance of hypothetico-deductive reasoning is caused by new reflective abstractions: the construction of new operations or operations based upon preceeding ones from the content of lower operations.

This abstraction reconstructs new operations from elements that are "reflected" from the lower level to the higher plane. By generalizing classification, the adolescent can construct the combinatorial system by classifying all the classifications (he constructs operations relating to antecedent operations). These operations upon operations provide the psychological novelty of hypothetico-deductive operations. For instance, see Item 15. These new operations permit us to combine the two heretofore unrelated systems containing inversion and reciprocity. The group of four transformations, INRC, does this on the formal level of operations.

5. Deductive Necessity

Since the reasoning encountered at the formal operations state of cognitive development is truly deductive in character we may now speak of the maturity of what we have been calling "logical necessity" and "self-evidence." This logical deduction is due to the deductive closure of the formal system under the 16 binary operations and the INRC group.

The set of all possibilities allows the adolescent to utilize deductive reasoning in a more mature sense, freed from materialistic constraints. Piaget notes that logically "... formal possibility is the required correlate of the notion of deductive necessity." (Piaget, 1958, 257)

To Piaget, any assertion that has a reference to empirical reality only, cannot be considered derived as a deductive necessity because the assertion would be true or false as it corresponds to the factual situation only. Whereas, a deduction that is logically derived correctly from a hypothesis or from data assumed hypothetically, is formally necessarily true independent of the hypothesized values.

The connection indicated by the words "if...then" (inferential implication) links a required logical consequence to an assertion whose truth is merely a possibility. This synthesis of deductive necessity and possibility characterizes the use of possibility in formal thought, as opposed to possibility-as-an-extension-of-the-actual-situation in concrete thought... (ibid., 257-8)

The synthesis of inversion and reciprocity in formal operations (operational reversibility) insures the deductive closure of the system and hence deductive necessity. The INRC group illustrates that the structures that formal thought elaborates are psychologically conserved. These structures are used, then, as "deductive instruments."

Regarding Item 14, the formal operational subject will now be deductively certain of the reversible compensations on the balance arm and will not have to actually perform the experiment. In contrast to the preceeding stage, the operational form is entirely dissociated from thought content and reasoning by implication, exclusion and disjunction, etc., is now possible. To have a combinational system presupposes a "structured whole" and thus a lattice structure characterized by the laws of of reciprocity.

6. Problems of the Model

While I have made considerable effort to describe Piaget's models of cognitive development and one of the mechanisms of transition from one stage of development to the next (i.e., reflective abstraction), it is not my intention to extensively analyze the controversial points. However, one particular criticism of Piaget's model of formal reasoning should be considered. There appears to be little in the way of criticism of the concrete operational model (Model I) and it has proven to be a useful model and framework within which to describe and interpret various behaviors of middle childhood.

Figure 24 is a compact illustration of Piaget's system of sixteen binary-operations as presented in Inhelder and Piaget (1958). I have adapted it from Ennis (1975). In the material implication interpretation of propositional logic, $p \rightarrow q = (p \cdot q) \vee (\bar{p} \cdot q) \vee (\bar{p} \cdot \bar{q})$ means that at least one of the three conjunctive elements holds true. Since $p \rightarrow q = \bar{p} \vee q$, the truth of q or the falsity of p establishes the material implication.

But as Ennis (1975), and Parsons (1960) before him, point out, Piaget appears to claim that the child establishes $p \rightarrow q$ only when he shows that all three conjunctions hold and that the conjunction $p \cdot \bar{q}$ (the inverse) does not. Piaget is not always clear on this matter (see Items 13 and 14 which are taken from Piaget). Of course, the standard interpretation of propositional logic will not tolerate the contradictions generated by Piaget's interpretation. Either p is true or not true, never both!

Piaget tries to circumvent these problems by treating these propositional symbols as propositional functions (propositions

| Piaget's Name and Number | Constructed Combination | Corresponding Number from Figure 19 | Piaget's Logical Shorthand |
|--|--|-------------------------------------|-------------------------------|
| 1. Complete affirmation | $p \cdot q \vee p \cdot \bar{q} \vee \bar{p} \cdot q \vee \bar{p} \cdot \bar{q}$ | 16 | $p * q$ |
| 2. Negation of complete affirmation | nothing | 1 | 0 |
| 3. Conjunction | $p \cdot q$ | 2 | $p \cdot q$ |
| 4. Incompatability | $p \cdot \bar{q} \vee \bar{p} \cdot q \vee \bar{p} \cdot \bar{q}$ | 15 | p / q |
| 5. Disjunction | $p \cdot q \vee p \cdot \bar{q} \vee \bar{p} \cdot q$ | 12 | $p \vee q$ |
| 6. Conjunctive negation | $\bar{p} \cdot \bar{q}$ | 5 | $\bar{p} \cdot \bar{q}$ |
| 7. Implication | $p \cdot q \vee \bar{p} \cdot q \vee \bar{p} \cdot \bar{q}$ | 14 | $p \quad q$ |
| 8. Nonimplication | $p \cdot \bar{q}$ | 3 | $p \cdot \bar{q}$ |
| 9. Reciprocal implication | $p \cdot q \vee p \cdot \bar{q} \vee \bar{p} \cdot \bar{q}$ | 13 | $q \quad p$ |
| 10. Negation of reciprocal implication | $\bar{p} \cdot q$ | 4 | $\bar{p} \cdot q$ |
| 11. Equivalence | $p \cdot q \vee \bar{p} \cdot \bar{q}$ | 8 | $p \quad q \text{ or } p = q$ |
| 12. Reciprocal exclusion | $p \cdot \bar{q} \vee \bar{p} \cdot q$ | 9 | $p \vee \vee q$ |
| 13. Affirmation of p | $p \cdot q \vee p \cdot \bar{q}$ | 6 | $p[q]$ |
| 14. Negation of p | $\bar{p} \cdot q \vee \bar{p} \cdot \bar{q}$ | 11 | $\bar{p}[q]$ |
| 15. Affirmation of q | $p \cdot q \vee \bar{p} \cdot q$ | 7 | $q[p]$ |
| 16. Negation of q | $p \cdot \bar{q} \vee \bar{p} \cdot \bar{q}$ | 10 | $\bar{q}[p]$ |

Figure 24. Piaget's 16 binary operations
Adapted from Ennis (1975)

containing variables). While he does not explicitly describe them as such in his 1958 work, Piaget does refer, apparently, to propositional functions in a later work.

Let us first describe them (in order to understand them) in the language of propositional functors.... Let there be a functor, for example $p \rightarrow q$, of which the normal disjunctive form is $p \vee \bar{p} \cdot q \vee \bar{p} \cdot \bar{q}$. (Piaget, 1966, 182)

Piaget's treatment can be made meaningful by the addition of existential and universal quantifiers. Thus, if at least one case of the conjunction is present, then the conjunction may appear in the disjunction ($p \cdot q$ means the existence of at least one case such that p and q). Thus, we should interpret the logical "or" symbol (\vee) as "and."

We can now easily interpret Figure 24. Choose the logical operator we are interested in from Column 1; the cases that must exist are shown in Column 2; those cases which must not exist (are excluded) are the ones missing from Column 2.

One of Ennis's most cogent objections to Piaget's model is that in some experimental situations it would be physically impossible to demonstrate the existence of certain cases and that this should not count against verifying an implication, for instance. He is, of course, correct in noting the impossibility of verifying the existence of certain cases, but Piaget might reply that: 1) either the reasoning model does not fully apply in these situations or that 2) what is required is the mere possibility of the existence of such a case rather than the actual existence.

As for 2), Ennis counters that it seems quite "unreasonable" to require the existence of a case for $\bar{p} \cdot q$ in order for $p \rightarrow q$ to hold. Also, suppose that it is physically impossible to verify the case $p \cdot q$, but we can assume that the case is at least possible. Then, Ennis

claims, anything is possible and the "Piagetian requirement" under the interpretation of the existence of possibilities becomes vacuous.

Another objection appears in the clash between the affirmation of p and q , simultaneously. To affirm p is to deny (according to Ennis's interpretation of Piaget's use of propositional logic) the existence of a case fitting $\bar{p}.q$ and to affirm q is to require the existence of at least one case fitting $\bar{p}.q$. Thus, to assert both p and q is to generate an inconsistency.

Also, to assert $p \rightarrow q$ is to deny the existence of a case fitting $p.\bar{q}$ and to assert $q \rightarrow p$ is to deny a case fitting $\bar{p}.q$ which is in turn required in $p \rightarrow q$. Notice that $q \rightarrow p$ also requires $p.\bar{q}$ thus making $p \rightarrow q$ and $q \rightarrow p$ incompatible. However, the biconditional, $p \leftrightarrow q$, requires the existence of at least one case fitting $p.q$ and $\bar{p}.\bar{q}$ but denies the existence of cases fitting $p.\bar{q}$ and $\bar{p}.q$.

Ennis also objects to the apparent overgeneralization caused by using propositional functions. Can the subject really justify inductive leaps from only a few cases to a generalization, say $p \rightarrow q$? Piaget might claim that the generalization only applies to the data already gathered about the problem but Ennis counters by asserting that we are then left unable to deal with future expectations. This seems to be a weak counter-argument for there is no need to go beyond the particular experiment at hand (Piaget does not do this in his 1958 work). However, to positively "establish" the non-existence (suspending ontological considerations) of the excluded case(s), say $p.\bar{q}$ in $p \rightarrow q$, is an inductive leap apparently justified only if the data can be exhaustively examined in a limited domain of discourse and data, of course.

While Piaget's formulation of the formal operational stage of development is not without apparent error, ambiguity or dispute, it is tollerably able to describe an important stage of cognitive development (see Elkind, 1961). Piaget's formalization is not meant to be a normative account of stages but a descriptive one.

The INRC group serves to adequately account for the notions of negation and reciprocity combined together within a single system. Piaget's discussion of these two forms of reversibility is formulated in terms of the INRC group as applied to propositions. Thus, he first translates the various actions of the system into propositions and then seeks to show that they conform to the notion of the four-group. I have followed this paradigm in my presentation of INRC.

While Piaget's attempts to describe the structures of cognitive development and their ontogenesis are far from faultless, they do at least provide a framework for empirical research in this area.

E. Children's Ability to "Handle" Propositional Logic: Piaget's View

Through all that has proceeded, it is possible to lose sight of the important notion that it is the character of the problem itself that will determine whether its solution will prompt formal reasoning or whether it can be solved in terms of an earlier operational level. For example, each of the experiemental situations in Inhelder and Piaget (1958) is an instance of a concrete situation whose solution cannot be reached by concrete operational structures. The situations are concrete presentations of a formal problem.

Let us see just how a purely verbal problem requiring formal reasoning for its solution can be reformulated so that it can be correctly answered by concrete operations. Consider the following two items.

Item 16 - Verbal Problem. "Mary is fairer than Alice; Mary is darker than Susan; which of the three is the fairest?"

We noted in the Illustrative Example for Grouping V that the concrete operational child is able to handle the two relations, "more x" and "less x" (serial ordering of asymmetrical transitive relationships) and it would appear that no other factors other than the coordination of these two relations are involved in this Item. However, this problem is not solved before the formal operations stage. Why?

Here, the subject must be able to extract the relational significance of the two statements from what Lunzer terms "the pictorial evocative connotation of the linguistic forms into which they are cast" (Lunzer, 1963, 587). Thus, the solution to the problem necessitates the examination of the formal and logical implications of the propositions, and not only their content. Lunzer believes that the solution turns on the "explicit" or "implicit" understanding of the equivalence: "Mary is fairer than Alice = Alice is darker than Mary." Apparently, children will tend to interpret "fairer" and "darker" in an absolute sense and will see Alice as "fair," Susan as "dark" and Mary as both "fair" and "dark." Thus, they erroneously conclude that Alice is the fairest.

Item 17 - Another Verbal Problem. Here is a version of Item 16 in a simpler form. "Mary is fairer than Alice; Alice is fairer than Susan; which of the three is fairest?" This form of the problem can be answered correctly by the concrete operational child. Why? Because, according to Piaget, it is sufficiently close to the concrete problem of serial ordering that the child can solve with concrete apparatus. The concrete serial ordering of shades should be no more difficult than setting lengths in series. Thus, if the verbal problem is "close" to the concrete problem, it can be solved by the concrete operational child. In Piaget's words:

However, this is not the whole problem, for all verbal thought is not formal and it is possible to get correct reasoning about simple propositions as early as the 7-8 year level, provided that these propositions correspond to sufficiently concrete representations. Even if the content of the complexion problem [Item 16] requires nothing more than serial order operations, the fact that it cannot be solved in exclusively verbal terms until several years after the child can solve it with the aid of physical props shows us that some other factor is at work here. If we consider the mental images involved in the problem we see how difficult it is for the subject to set up the data in his own mind (because only the relations are given). The result is that the subject is unable to translate the data into representational imagery and has to formulate them in exclusively hypothetical terms if he is to see the necessary consequences. (Inhelder and Piaget, 1958, 252)

Thus, to claim, as some do (see Ennis 1975, 1976) that only formal operational children can deal with verbal propositions, is incorrect. We see that both concrete and formal operational children can solve certain kinds of problems expressed in verbal propositional form. But can a child "handle" (reason correctly about) verbal propositional logic statements of the type: "If Mary is in school, then Paul is home. Mary is in school. Therefore, Paul is home?" Clearly, these propositional statements together form a logical argument whose conclusion follows deductively from the premises (logical necessity). Is it only the formal operational child who is able to handle this logic? If so, to what extent?

Piaget cautions against using linguistic criteria alone to determine whether a child is reasoning with formal operations (see for example, Inhelder and Piaget, 1958, 279) but offers no guidance as to whether concrete and/or formal operational children can reason correctly with conditional logic propositions (i.e., understand the principles of conditional logic). Recall that it is the character of the problem that will prompt the need for a particular level of reasoning. Thus, this seems to be a question that admits of an empirical answer - theory can take us only so far.

F. Summary

In this report I have outlined Jean Piaget's theory of the ontogenesis of logical necessity and the concomitant notion of self-evidence. We saw how the child progresses by the processes of adaptation and reflective abstraction from a neonate unable to distinguish self from the world, to an active agent capable of interiorizing actions into operations, to an individual able to operate upon concrete objects, to

the stage where she can operate upon concrete operations (formal operations). Along this developmental path can be seen the successive levels of logical necessity. I have illustrated formal models that describe the concrete and formal operational stages and have touched upon some of the problems associated with one of the models.

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