Proof of the Uniqueness Conjecture of Visible Opposing Same Color Pairs

Conjecture: If a stacked tower of 4 cubes has a unique solution, no cube in the tower may have an opposing pair of the same color on the visible sides of the tower.

Definitions:

Valid Transformation: A transformation of a cube is considered a valid transformation if the transformation is performed by flipping or rotating a cube by a multiple of 90 degrees, defined from the cube's starting position.

Configuration: If two or more sets of valid transformations cause a cube to have an identical positional orientation, that orientation is considered a configuration of the cube.

Solution: A stack of 4 cubes is considered a solution if each side of the tower generated by stacking the 4 cubes contains 4 different colors.

Unique Solution: For a given set of 4 cubes, a solution is considered a unique solution if it is not equivalent to any other solution.

Rotational Equivalence Class: A solution is considered rotationally equivalent to another solution if it can be generated by simultaneously rotating all 4 cubes of the solution to the same degree and in the same direction. Solutions which are rotationally equivalent belong to the same rotational equivalence class.

Flip Equivalence Class: A solution is considered flip equivalent to another solution if it can by generated by flipping all 4 cubes of the tower 180 degrees in the same direction along the same axis. Solutions which are flip equivalent belong to the flip equivalence class.

Swap Equivalence Class: A solution is considered swap equivalent to another solution if it can be generated by swapping one or more cubes on one of the 4 levels of the tower with cubes on another level of the tower. Solutions which are swap equivalent belong to the swap equivalence class.

Proof:

- If a tower has a unique solution, each cube in the tower must have a unique configuration.
- Each cube configuration in a solution tower must have a necklace of four colors around the sides of the cube. (The colors on the necklace will be referred to as beads.)
- Therefore, if a tower has a unique solution, each necklace formed by the sides of a cube must be unique.
- Let n be an integer such that $1 \leq n \leq 4$. Assume that n necklaces formed by the sides of the cubes in a solution tower contain at least one pair of opposing beads of the same color separated on the necklace by 1 bead on both sides.
- Let us choose 1 of the n necklaces at random and rotate it 180 degrees.
- We can partition the results of this rotation into 2 outcomes: outcomes in which the rotation creates a new solution, and outcomes in which it does not.

- For outcomes in which a new solution is created by the rotation, the original solution is not unique.
- For outcome in which a new solution is not formed, the 2 beads which are not included in the same color opposing pair swap positions.
- We may flip the necklace such that these two beads swap back to their original position while also leaving the same color opposing pair in place.
- This flip will create a new solution. Therefore, the original solution is not unique.
- Therefore, for all outcomes of this 1 necklace, the tower will not have a unique solution.
- If there are *n* necklaces that contain at least one pair of same color beads opposing each other, it will *always* be possible to choose *only one* of the cubes and perform the preceding actions on the cube. Therefore, if *any* cube on a tower has an opposing same color pair on the visible sides of the tower's solution, that solution is not unique.

Q.E.D.